Sets, Functions, and Relations
Sets are unordered collections of objects.

- If a set is finite, we can represent it by listing all its elements in curly braces:

  \[ A = \{42, 9, 22, 78\} \]

- We can also specify a set by establishing a pattern and using ellipses:

  \[ B = \{1, 3, 5, 7, \ldots\} \]

- But this can be ambiguous, so we prefer set builder notation:

  \[ c = \{x \mid x \text{ is a prime number}\} \]
we use '∈' to express set membership.

Michael ∈ \{x | x is faculty at IIT\}

W ∈ \{c | c is a letter in the English alphabet\}

we can also use this in set builder notation:

\[ C = \{i | i \text{ is a positive integer}\} \]
\[ D = \{j \in C | 100 \leq j \leq 200\} \]

given predicate P(x), we define its truth set over domain D:

\[ T_p = \{x \in D | P(x)\} \]
\( \emptyset \) or \( \{\} \) denote the empty set.

Important: \( \{\emptyset\} \) is not the empty set!

(it is a set containing one element, which happens to be the empty set — also: \( \{\{\}\}\) )
T/F?

\[ a \in \{a, b, c\} \quad \{a\} \notin \{a,b,c\} \]
\[ a \in \emptyset \quad \emptyset \notin \{a\} \]
\[ \emptyset \in \{\} \quad \emptyset = \{\} \]
\[ 2 \in \{w \mid 6 \not\exists x \mid x \text{ is divisible by } w\} \]
Some fixed names we use for sets of numbers:

- **N**: natural numbers \{0, 1, 2, 3, \ldots \}
- **Z**: integers \{-\ldots, -2, -1, 0, 1, 2, \ldots \}
- **R**: real numbers
- **Q**: rational numbers = \{ \frac{p}{q} | p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0 \}

\(\mathbb{Z}^+ / \mathbb{Z}^- / \mathbb{R}^+ / \mathbb{R}^- / \mathbb{Q}^+ / \mathbb{Q}^-\) are the sets of positive/negative integers, reals, rationals

E.g., \(\{x \in \mathbb{R} | x^2 < 9 \}\)
we can restrict the domain of a quantifier using sets:

e.g., universal quantification $\forall x P(x)$ for domain $D$:

$$\forall x \in D (P(x))$$

$$\equiv \forall x (x \in D \rightarrow P(x))$$

e.g. existential quantification $\exists x P(x)$ for domain $D$:

$$\exists x \in D (P(x))$$

$$\equiv \exists x (x \in D \land P(x))$$
sets are equal if they contain the same distinct elements. (order doesn't matter, duplicates don't matter!)

equal or not?
\[
\{1, 2, 3, 4 \} = \{4, 3, 2, 1 \}
\]
\[
\{1, 1, 2, 3, 2, 2, 4 \} = \{1, 2, 3, 4 \}
\]
\[
\{x \in \mathbb{N} \mid x < 4 \} = \{x \in \mathbb{Z} \mid 0 \leq x < 4 \}
\]
\[
\{x \in \mathbb{Z} \mid x^2 < x \} = \{x \in \mathbb{R} \mid x^2 < 0 \}
\]
The cardinality of a finite set $A$, denoted $|A|$, is the number of distinct elements in $A$.

$$|\{0, 1, 2, 3\}| = 4$$

$$|\emptyset| = 0$$

$$|\{2, 2, 3, 1, 2, 3\}| = 3$$

$$|\{x \mid x \text{ is prime and } x < 10\}| = 4$$
A finite set is countable.

An infinite set is countable if we can find a one-to-one correspondence between its elements and the natural numbers.

(CS intuition: countable if we can "index" its elements)
is the set of even numbers \( \{0, 2, 4, \ldots \} \) countable?

\[
N = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad \ldots
\]

\[
evens = 0 \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad 14 \quad 16 \quad \ldots
\]

there is a one-to-one correspondence between \( \mathbb{N} \) and evens.

Ans. Yes!
is \( \mathbb{Z} \) (integers \{-3, -2, -1, 0, 1, 2, 3, \ldots \}) countable?

\[
N = \{0, 1, 2, 3, 4, 5, 6, \ldots \} = \{0, 1, -1, 2, -2, 3, -3, \ldots \}
\]

mapping function: \( f(n) = \begin{cases} \frac{-n}{2} & \text{if } n \text{ is even} \\ \frac{(n+1)}{2} & \text{if } n \text{ is odd} \end{cases} \)

Rewrite \( \mathbb{Z} \) as \( \{0, 1, -1, 2, -2, 3, -3, \ldots \} \)

Ans. Yes!
is $\mathbb{Q} = \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0 \right\}$ countable?

Ps I 2 3 4 5 ... 
Gs,
o
y '4, 51, 641, 5h
22424427424/2 Ans.
Yes!
3 343843 3/34/3
4944" 2/4 3/4
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-
I
is \( \mathbb{R} \) (real numbers) countable?

- Let's assume yes
- Consider all reals between 0 and 1:

\[
\begin{align*}
0 & \rightarrow 0.\overline{a_0 a_1 a_2 a_3} \\
1 & \rightarrow 0.\overline{a_1 a_0 a_2 a_3} \\
2 & \rightarrow 0.\overline{a_2 a_0 a_1 a_3} \\
3 & \rightarrow 0.\overline{a_3 a_0 a_1 a_2} \\
& \vdots \\
N & \rightarrow 1.\overline{0}0000 \ldots
\end{align*}
\]

Let's pick digits on diagonal and make a new real where its digits differ from these.

E.g., \( b_{ij} = (a_{ij} + 1) \mod 10 \)

This new real is not in our list!

\( \therefore \) \( \mathbb{R} \) is NOT countable.
the power set of $S$, $\mathcal{P}(S)$, is the set of all subsets of $S$.

$\mathcal{P}(\{a, b, c\}) = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

$\mathcal{P}(\{a\}) = \{ \emptyset, \{a\}\}$

$\mathcal{P}(\emptyset) = \{ \emptyset \}$

For set $S$ with $|S| = n$, $|\mathcal{P}(S)| = 2^n$
Set operations, given sets A and B:

- **Union**: \(A \cup B\)
  \[A \cup B = \{x \mid x \in A \lor x \in B\}\]

- **Intersection**: \(A \cap B\)
  \[A \cap B = \{x \mid x \in A \land x \in B\}\]

- **Difference**: \(A - B\) or \(A \setminus B\)
  \[A - B = \{x \mid x \in A \land x \notin B\}\]

- **Complement**: \(\overline{A}\)
  \[\overline{A} = \{x \in U \mid x \notin A\}\]
  \(\Rightarrow\) “universal” set
**TABLE 1** Set Identities.

<table>
<thead>
<tr>
<th>Identity</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \cap U = A$</td>
<td>Identity laws</td>
</tr>
<tr>
<td>$A \cup \emptyset = A$</td>
<td></td>
</tr>
<tr>
<td>$A \cup U = U$</td>
<td>Domination laws</td>
</tr>
<tr>
<td>$A \cap \emptyset = \emptyset$</td>
<td></td>
</tr>
<tr>
<td>$A \cup A = A$</td>
<td>Idempotent laws</td>
</tr>
<tr>
<td>$A \cap A = A$</td>
<td></td>
</tr>
<tr>
<td>$\overline{(A)} = A$</td>
<td>Complementation law</td>
</tr>
<tr>
<td>$A \cup B = B \cup A$</td>
<td>Commutative laws</td>
</tr>
<tr>
<td>$A \cap B = B \cap A$</td>
<td></td>
</tr>
<tr>
<td>$A \cup (B \cup C) = (A \cup B) \cup C$</td>
<td>Associative laws</td>
</tr>
<tr>
<td>$A \cap (B \cap C) = (A \cap B) \cap C$</td>
<td></td>
</tr>
<tr>
<td>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$</td>
<td>Distributive laws</td>
</tr>
<tr>
<td>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$</td>
<td></td>
</tr>
<tr>
<td>$\overline{A \cap B} = \overline{A} \cup \overline{B}$</td>
<td>De Morgan’s laws</td>
</tr>
<tr>
<td>$A \cup \overline{B} = \overline{A} \cup B$</td>
<td></td>
</tr>
<tr>
<td>$A \cap (A \cup B) = A$</td>
<td>Absorption laws</td>
</tr>
<tr>
<td>$A \cap (A \cup B) = A$</td>
<td></td>
</tr>
<tr>
<td>$A \cup \overline{A} = U$</td>
<td>Complementation laws</td>
</tr>
<tr>
<td>$A \cap \overline{A} = \emptyset$</td>
<td></td>
</tr>
</tbody>
</table>

**proof**

$$A \cup B = \overline{A} \land \overline{B} :$$

$$\overline{A \cup B} = \{x \mid x \notin (A \cup B)\}$$

$$= \{x \mid \neg (x \in (A \cup B))\}$$

$$= \{x \mid \neg (x \in A \lor x \in B)\}$$

$$= \{x \mid x \notin A \land x \notin B\}$$

$$= \{x \mid x \in \overline{A} \land x \in \overline{B}\}$$

$$= \overline{A} \land \overline{B}$$
\[ A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k = \bigcup_{i=1}^{k} A_i \]

\[ A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_k = \bigcap_{i=1}^{k} A_i \]
we say \( A \) is a \underline{subset} of \( B \) \((A \subseteq B)\), and \( B \) is a \underline{superset} of \( A \) \((B \supseteq A)\), iff every element in \( A \) is also in \( B \). i.e.,

\[
\forall x (x \in A \rightarrow x \in B)
\]

We say \( A \) is a \underline{proper subset} of \( B \) \((A \subset B)\) iff \( A \) is a subset of \( B \) but \( A \neq B \). i.e.,

\[
\forall x (x \in A \rightarrow x \in B) \land \exists x (x \in B \land x \notin A)
\]
sets are unordered, but sometimes order matters!

an $n$-tuple is an ordered collection of $n$ elements

$(1, 2), (2, 1), (1, 1, 2), (2, 2, 1, 2, 1)$ are all distinct!

"ordered pairs"
the Cartesian product of two sets $A$ and $B$ is:

$$A \times B = \{(a,b) \mid a \in A \land b \in B\}$$

e.g., $\{1, 2, 3\} \times \{p, q\} = \{(1,p), (1,q), (2,p), (2,q), (3,p), (3,q)\}$

$$\{x, y, z\} \times \emptyset = \emptyset$$

$$\{x, y, z\} \times \{\emptyset\} = \{(x,\emptyset), (y,\emptyset), (z,\emptyset)\}$$
the Cartesian Product of sets $A_1, A_2, \ldots A_n$ is defined as:

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

e.g. $\{a, b\} \times \{c, d\} \times \{e, f\} = \{(a, c, e), (a, c, f), (a, d, e), (a, d, f), (b, c, e), (b, c, f), (b, d, e), (b, d, f)\}$

we also write $A^2$ for $A \times A$, $A^3$ for $A \times A \times A$, i.e.,

$$A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } i = 1, 2, \ldots, n\}$$

e.g. $\{a, b, c\}^2 = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$
Note: $A \times B \neq B \times A$ (unless $A$ or $B$ is $\emptyset$),
and $A \times B \times C \neq (A \times B) \times C$!

\[ e.g. (\{a, b\} \times \{c, d\}) \times \{e, f\} \]
\[ = \{(a, c), (a, d), (b, c), (b, d)\} \times \{e, f\} \]
\[ = \{(a, c), (a, d), (b, c), (b, d), (a, c), (a, d), (b, c), (b, d)\} \]
A relation from set A to set B is a subset of $A \times B$.

E.g., given $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, write down a relation from A to B:

$$G = \{(1, b), (3, c), (2, a), (2, c)\}$$

We say that 3 is related to c through $G (3 Rc)$, 3 is not related to b through $G (3 Rb)$.
A relation on a set $A$ is a relation from $A$ to $A$ (i.e., a subset of $A \times A$).

e.g., given $A = \{1, 2, 3, 4\}$, write down a relation on $A$.

$$R = \{(1,2), (1,3), (2,4), (4,1)\}$$
A function \( f \) from set \( A \) to set \( B \) is a relation from \( A \) to \( B \) where there is exactly one ordered pair \((a, b)\) for each \( a \in A\).

- We write \( f : A \rightarrow B \) (\( f \) maps set \( A \) to set \( B \)) and \( f(a) = b \) (\( f \) assigns element \( b \) to element \( a \)).

- \( A \) is the domain of \( f \), and \( B \) is the codomain.

- The set of all elements of \( B \) assigned by \( f \) is called the range.