Counting + Combinatorics

CS 330 : Discrete Structures

Counting is fundamental to answering many problems that arise in discrete mathematics.

E.g., how many operations are required to carry out a given algorithm on some input?

how many guesses do we need to brute force a password / key?

how many different arrangements of pieces does a game-playing algorithm need to consider?

Combinatorics is a branch of mathematics that deals with counting and arranging things!

e.g., combinations, permutations, enumeration

Important: we don't typically want to enumerate all the things in order to count them!

sometimes "counting" can use quite a bit of mathematical cleverness!

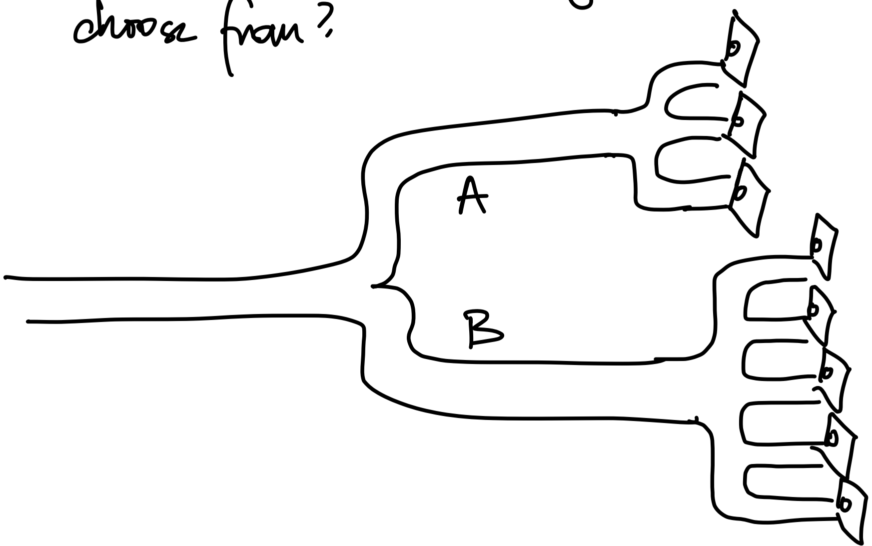
Basic combinatorial techniques:

1. Sum rule
2. Product rule
3. Subtraction rule
4. Division rule

Sum rule

if we must choose an element from either set A or B, and A and B are disjoint (i.e., $A \cap B = \emptyset$), then there are $|A| + |B|$ choices in total.

e.g., if a dungeon gives us a choice to go down one of two paths A or B, and A ends in 3 doors and B ends in 5, how many different doors can we choose from?



$3+5=8$

Product rule

- if we must choose an element from both set A and B (i.e., one from each), then there are $|A| \times |B|$ choices in total.

* the choices must be <u>independent</u>. i.e., regardless of the element we choose in A, there are still $|B|$ choices in B, and vice versa.

e.g., a proposed license plate numbering system consists of 3 letters followed by 3 digits. How many possible license plates are there?

A : pick letter 1 — 26 choices
B : " " 2 — " "
C : " " 3 — " "
D : " digit 1 — 10 choices
E : " " 2 — " "
F : " " 3 — " "

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17{,}576{,}000$$

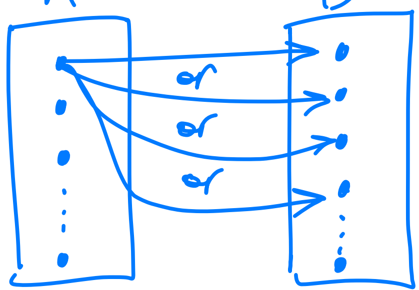
e.g., a travelling salesman wants to visit all 50 states.
how many different ways (i.e., distinguished by visitation order) can they do this?

1st state = 50 choices
2nd " = 49 choices
3rd " = 48 choices
⋮

$50 \times 49 \times 48 \times \ldots \times 1 = 50!$

$\approx 3.04 \times 10^{64}$

each specific ordering is called a "permutation" e.g., how many functions are there from set A to set B?



$|A| = n_A$ $|B| = n_B$

- for each element of A, we have n_B choices
- we must choose independently for all values of A
- $\underbrace{n_B \times n_B \times \ldots \times n_B}_{n_A} = n_B^{n_A}$ ways

e.g. how many one-to-one function are there from A to B?

- for a_1, we have n_B choices; for a_2, $n_B - 1$ choices, ...
- $n_B \times (n_B - 1) \times (n_B - 2) \times \ldots \times (n_B - (n_A - 1))$

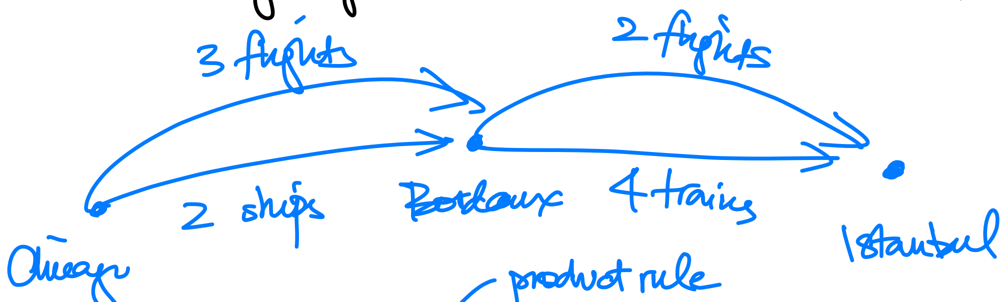
choices taken by preceding values in A

e.g., I want to travel to Istanbul.

There are 3 flights and 2 ships to Bordeaux

From Bordeaux, there are 2 flights and 4 trains to Istanbul.

How many ways are there to travel to Istanbul via Bordeaux?



product rule

$$\underbrace{(3+2)}\times\underbrace{(2+4)} = 5 \times 6 = 30$$

sum rule

e.g., how many different top-ten rankings exist for a list of 100 songs?

$$100 \times 99 \times 98 \times 97 \times 96 \cdots \times 91 = \frac{100!}{90!}$$

an r-permutation of set A is an ordered collection of r elements drawn from A.

given a set of n elements, and $1 \leq r \leq n$, how many distinct r-permutations of the set are there?

$$P(n,r) = \frac{n!}{(n-r)!}$$

* each permutation contains no repeated elements!

e.g., how many different sequences of 5 rolls are there given a 6-sided die?

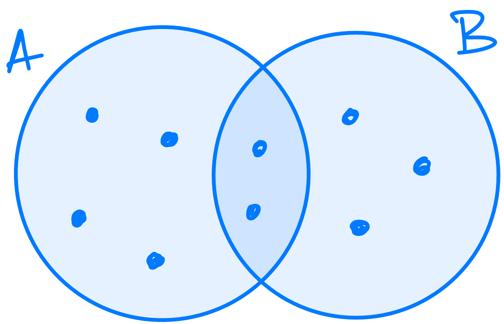
$$6 \times 6 \times 6 \times 6 \times 6 = 6^5$$

given a set of n elements, and $1 \leq r \leq n$, how many distinct r-permutations of the set are there, if repetitions are allowed?

$$= n^r$$

Subtraction rule (aka. inclusion-exclusion)

if we must choose an element from either set A or B, where A and B are overlapping (i.e., $A \cap B \neq \emptyset$), then there are $|A| + |B| - |A \cap B|$ choices in total.



total unique choices

$= |A| \, (6) + |B| + 5 - |A \cap B| \, (2)$

$= 11 - 2 = 9$

e.g., given 100 CS majors, 50 CPE majors,

how many total students in these two departments?

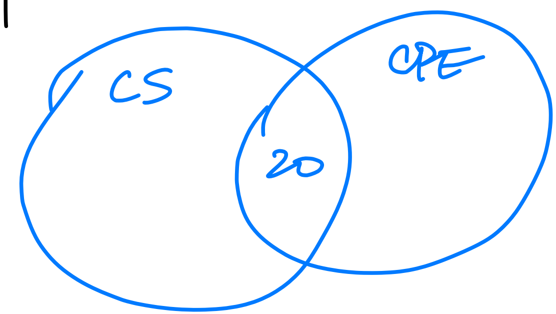
100 + 50 = 150 ?

* only if sets of CS and CPE majors are disjoint sets!

e.g., given 100 CS majors, 50 CPE majors

how many total students in these two departments?

also given: 20 CS/CPE majors



$$|CS \cup CPE| = |CS| + |CPE| - |CS \cap CPE|$$

$$= 100 + 50 - 20 = 130$$

e.g., given 100 CS majors, 50 CPE majors, 30 EE majors,

how many total students in these three departments?

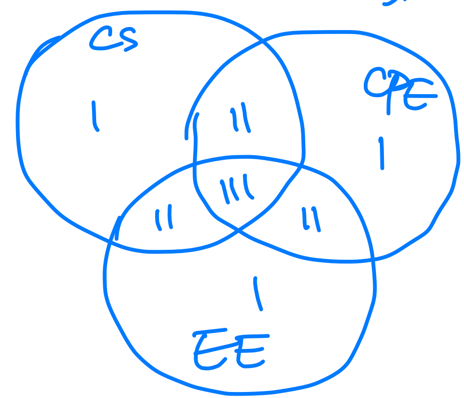
also given: 20 CS/CPE majors

15 CS/EE majors

10 CPE/EE majors

5 CS/CPE/EE majors

(times counted in $|CS \cup CPE \cup EE|$ shown)



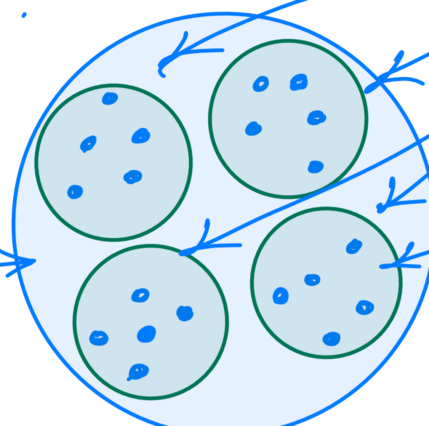
$$
\begin{aligned}
|CS \cup CPE \cup EE| &= |CS| + |CPE| + |EE| \\
&\quad - |CS \cap CPE| - |CS \cap EE| - |CPE \cap EE| \\
&\quad + |CS \cap CPE \cap EE| \\
&= 100 + 50 + 30 - 20 - 15 - 10 + 5 = 140
\end{aligned}
$$

Division rule

if set A is the union of n non-overlapping sets, each containing d elements, then $|A| = n \cdot d$, and $n = \frac{|A|}{d}$

how is this useful?

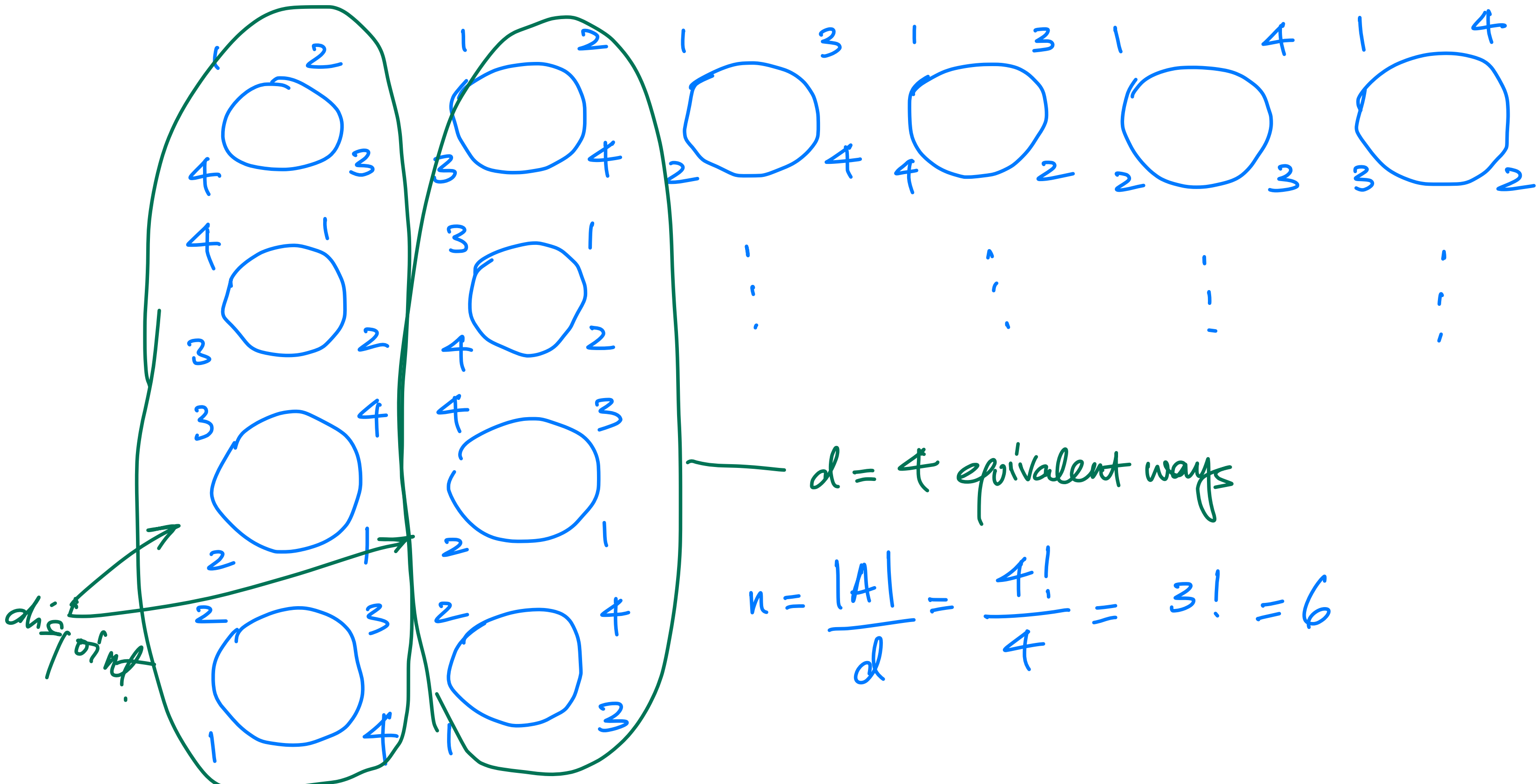
n total distinct elements



d elements that are effectively equivalent

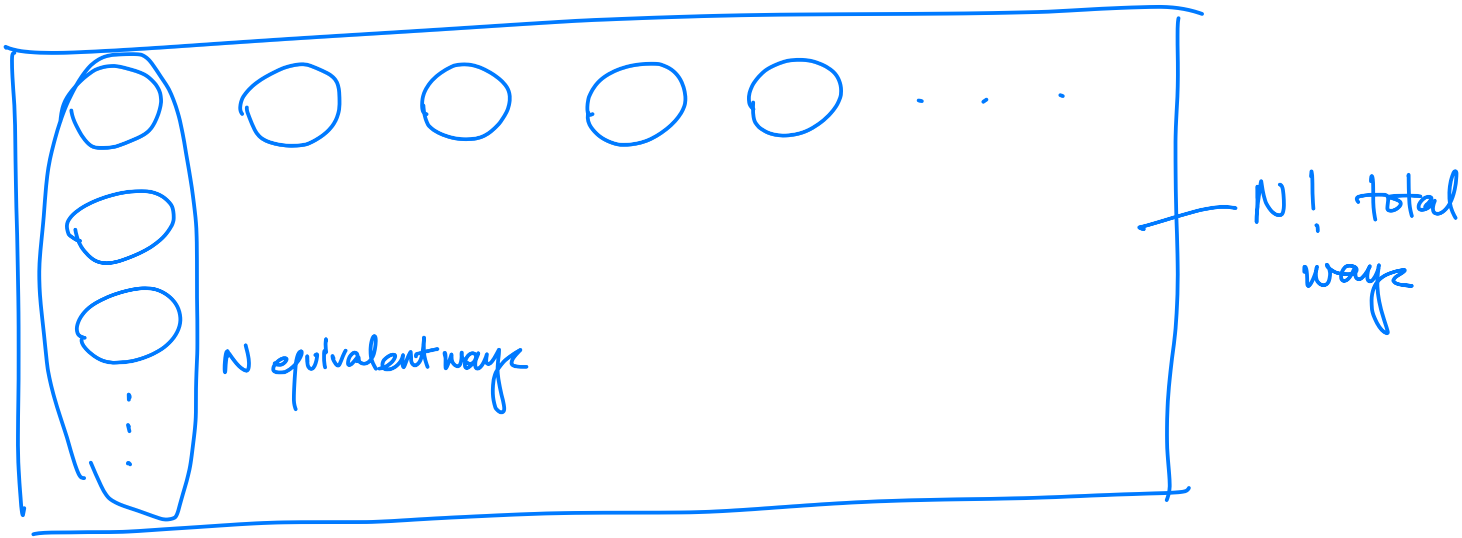
A = all elements under consideration

e.g., how many ways are there of seating 4 guests at a round table?



$$n = \frac{|A|}{d} = \frac{4!}{4} = 3! = 6$$

e.g., how many ways are there of seating N guests at a round table?



$$\text{total ways} = \frac{N!}{N} = (N-1)!$$

e.g., how many different groups of 4 students can be formed from a 100-student class?

4-permutations $= 100 \times 99 \times 98 \times 97$

– but order doesn't matter w/in group!

equivalent ways of ordering 4 students $= 4! = 24$

by division rule, # different groups $= \dfrac{100 \times 99 \times 98 \times 97}{24}$

an r-combination of set A is a subset of A w/ r elements.

how many r-combinations exist for an n-element set?

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$ ✱ without repetition!

"n choose r"

aka "binomial coefficient"

Some special cases:

$$\binom{n}{0} = \frac{n!}{n!\,0!} = 1 \qquad \emptyset \text{ is the only size } 0 \text{ subset}$$

$$\binom{n}{1} = \frac{n!}{(n-1)!\,1!} = n \qquad n \text{ subsets of size } 1$$

$$\binom{n}{n} = \frac{n!}{0!\,n!} = 1 \qquad 1 \text{ subset of size } n \text{ (the set itself)}$$

$$\binom{n}{n-1} = \frac{n!}{1!\,(n-1)!} = n \qquad n \text{ subsets of size } n-1$$

e.g. what are the coefficients in the expansion of $(x+y)^3$?

$$(x+y)(x+y)(x+y)$$

$$=(x^2+2xy+y^2)(x+y)$$

$$=(x^3+x^2y+2x^2y+2xy^2+xy^2+y^3)$$

$$=(x^3+3x^2y+3xy^2+y^3)$$

e.g. what are the coefficients in the expansion of $(x+y)^4$?

$$(x+y)^4 = ?\,x^4 + ?\,x^3y + ?\,x^2y^2 + ?\,xy^3 + ?\,y^4 \quad \ldots$$

how many ways to choose 4 x's from these factors? $=1$

how many ways to choose 3 x's? $=\binom{4}{3}=4$

how many ways to choose 2 x's? (or 2 y's) $=\binom{4}{2}=6$

$$\underbrace{(x+y)(x+y)(x+y)(x+y)}$$

Binomial Theorem

$$\forall n \in \mathbb{N}, \quad (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

e.g. what are the coefficients in the expansion of $(x+y)^4$?

$$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

$$= 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$

Prove $\forall n \geq 0$ $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

① binomial theorem: $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}$

② combinatorial: consider # of subsets of an n-element set.

$$\left.\begin{array}{llll}
- \text{ \# of} & \text{0-element} & \text{subsets} & = \binom{n}{0} \\
- \text{ "} & \text{1- "} & \text{"} & = \binom{n}{1} \\
- \text{ "} & \text{2- "} & \text{"} & = \binom{n}{2} \\
\quad \vdots & & & \\
- \text{ "} & \text{n- "} & \text{"} & = \binom{n}{n}
\end{array}\right\} \text{sum} = \sum_{k=0}^{n} \binom{n}{k}$$

i.e., $2^n = \sum_{k=0}^{n} \binom{n}{k}$

- we also know that set of n elements has 2^n subsets

Combinatorial proof techniques:

1. **Double counting proof**: show that two expressions e_1 and e_2 are just different ways of counting the same set S. Since $|S| = e_1$ and $|S| = e_2$, $e_1 = e_2$.

2. **Bijective proof**: prove the existence of a bijective function $f: A \to B$, which shows $|A| = |B|$. If we can count B, then we can also count A.

prove: $\binom{n}{r} = \binom{n}{n-r}$ when $0 \leq r < n$

e.g. $\binom{6}{2} = \binom{6}{4} - \frac{6!}{(6-2)!\,2!} = \frac{6!}{(6-4)!\,4!}$

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \frac{n!}{r!\,(n-r)!} = \frac{n!}{(n-(n-r))!\,(n-r)!} = \binom{n}{n-r}$$

(algebraic proof)

prove: $\binom{n}{r} = \binom{n}{n-r}$ when $0 \leq r < n$

combinatorial:

- consider a set S w/ n elements, and the set R, which contains all subsets of S w/ r elements; $|R| = \binom{n}{r}$
- the function $f: R \to R'$, where $f(a) = \overline{a}$, is a bijection between the subsets of S w/ r elements and the subsets of S w/ $n-r$ elements; $|R'| = \binom{n}{n-r}$
- since $f: R \to R'$ is a bijection, $|R| = |R'|$, and $\binom{n}{r} = \binom{n}{n-r}$

prove $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ for $0 \le k < n$

- consider trying to form a team of k players from a roster of n $= \binom{n}{k}$
- consider trying out for a team w/ k spots against $n-1$ other players
  - either ① you are selected, along w/ $k-1$ other players $= \binom{n-1}{k-1}$

    or ② you are not selected, so k players are chosen $= \binom{n-1}{k}$

    these are disjoint!

so $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

"Pascal's Triangle" identity

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5} \\
\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6} \\
\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7} \\
\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}
\end{array}
$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}
$$

e.g., given A = ways to select a dozen donuts from 5 varieties

B = all 16-bit sequences w/ exactly four 1's,

show a bijection from $A \to B$.

e.g. of A: 000 00 00 0 0000

chocolate glazed strawberry plain cruller

e.g. of B: 000 1 00 1 00 1 0 1 0000

there is a bijection from $A \to B$! so $|A| = |B|$

e.g., given A = ways to select a dozen donuts from 5 varieties

B = all 16-bit sequences w/ exactly four 1's ,

$$\underbrace{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \end{matrix}}_{\text{need to pick 4 bits to turn "on"}}$$

e.g., $\{6, 8, 12, 16\} \longrightarrow 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1$

i.e., there is a bijection from the set of n-bit sequences w/ m 1's to the set of m-element subsets of an n-element set.

$$\#\text{ of } n\text{-bit sequences w/ } m \text{ 1's} = \binom{n}{m}$$

e.g., given A = ways to select a dozen donuts from 5 varieties

B = all 16-bit sequences w/ exactly four 1's ,

# of n-bit sequences w/ m 1's $= \binom{n}{m}$

# of ways to select d donuts from t types $= \binom{d+t-1}{t-1} \underbrace{= \binom{d+t-1}{d}}_{\text{by } \binom{n}{r} = \binom{n}{n-r}}$

$$|A| = |B| = \binom{16}{4}$$

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

$P(n,r)$

$C(n,r) = \binom{n}{r}$

r donuts from n types

$C(n+r-1, n-1)$

$= \binom{n+r-1}{n-1}$

e.g., how many different strings can we make from permuting characters in "BOOKKEEPER"?

- 10 chars, drawn from: 1 B, 2 O's, 2 K's, 3 E's, 1 P, 1 R

"indistinguishable" elements

- bijection to the # of ways we map char positions to

$$\left(\{\underline{1}\}, \{\underline{2}, \underline{3}\}, \{\underline{4}, \underline{5}\}, \{\underline{6}, \underline{7}, \underline{9}\}, \{\underline{8}\}, \{\underline{10}\}\right) \text{ (e.g).}$$

$$\binom{10}{1} \cdot \binom{9}{2} \cdot \binom{7}{2} \cdot \binom{5}{3} \cdot \binom{2}{1} \cdot \binom{1}{1}$$

"permutations of indistinguishable objects"

# of permutations of n elements, given k_1 indistinguishable elements of type 1, k_2 indistinguishable elements of type 2, ..., and k_m indistinguishable elements of type m, is:

$$\binom{n}{k_1} \cdot \binom{n-k_1}{k_2} \cdot \ldots \cdot \binom{n-k_1-\ldots-k_{m-1}}{k_m}$$

$$= \frac{n!}{\cancel{(n-k_1)!}\,k_1!} \cdot \frac{\cancel{(n-k_1)!}}{\cancel{(n-k_1-k_2)!}\,k_2!} \cdot \ldots \cdot \frac{\cancel{(n-k_1-\ldots-k_{m-1})!}}{\cancel{(n-k_1-\ldots-k_{m-1}-k_m)!}\,k_m!}$$

0

$$= \frac{n!}{k_1!\,k_2!\ldots k_m!}$$

e.g., how many different strings can we make from permuting characters in "BOOKKEEPER"?

$$\frac{10!}{1!\,2!\,2!\,3!\,1!\,1!} = 15,1200$$

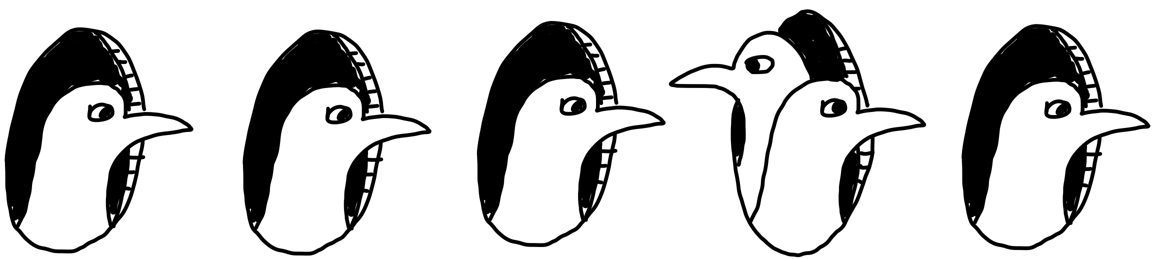
e.g.: you are guessing a video game code that consists of 2 ups, 2 downs, 2 lefts, and 2 rights. How many possible codes exist?

$$\frac{8!}{(2!)^4} = 2,520$$

e.g., how many socks must you pull out of a drawer containing red, green, and blue socks before you find a matching pair?

Pigeonhole Principle

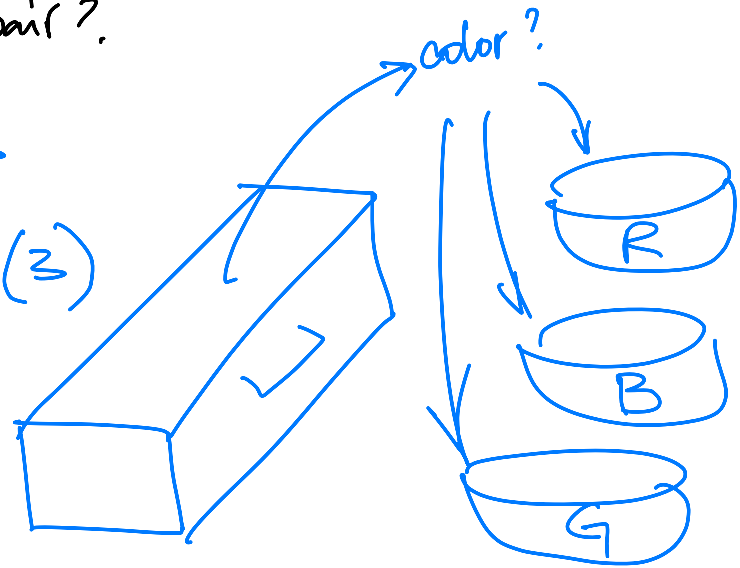
"if there are n pigeons occupying m holes, and $n > m$, then there must be at least one hole w/ two pigeons



e.g., how many socks must you pull out of a drawer containing red, green, and blue socks before you find a matching pair?

socks = pigeons

colors = holes (3)



generalized Pigeonhole Principle:

for sets A, B, if $|A| > k \cdot |B|$, then every function $f: A \to B$ maps at least $k+1$ elements of A to the same element of B.

e.g., at least how many people in Chicago have the same # of hairs on their head? (ignoring bald people)

pop. of Chicago $\approx$ 2.5 million (say, > 2 million non-bald)

max # of hairs on a head = 200,000

let A be set of non-bald chicagoans

B be $\{h \mid 0 < h \leq 200{,}000\}$

$f: A \to B$, map a person to # of hairs on their head

$|A| > 10 \cdot |B|$ — at least 11 people!

0020480135385502964448038	3171004832173501394113017
5763257331083479647409398	8247331000042995311646021
0489445991866915676240992	3208234421597368647019265
5800949123548989122628663	8496243997123475922766310
1082662032430379651370981	3437254656355157864869113
6042900801199280218026001	8518399140676002660747477
1178480894769706178994993	3574883393058653923711365
6116171789137737896701405	8543691283470191452333763
1253127351683239693851327	3644909946040480189969149
6144868973001582369723512	8675309258374137092461352
1301505129234077811069011	3790044132737084094417246
6247314593851169234746152	8694321112363996867296665
1311567111143866433882194	3870332127437971355322815
6814428944266874963488274	8772321203608477245851154
1470029452721203587686214	4080505804577801451363100
6870852945543886849147881	8791422161722582546341091
1578271047286257499433886	4167283461025702348124920
6914955508120950093732397	9062628024592126283973285
1638243921852176243192354	4235996831123777788211249
6949632451365987152423541	9137845566925526349897794
1763580219131985963102365	4670939445749439042111220
7128211143613619828415650	9153762966803189291934419
1826227795601842231029694	4815379351865384279613427
7173920083651862307925394	9270880194077636406984249
1843971862675102037201420	4837052948212922604442190
7215654874211755676220587	9324301480722103490379204
2396951193722134526177237	5106389423855018550671530
7256932847164391040233050	9436090832146695147140581
2781394568268599801096354	5142368192004769218069910
7332822657075235431620317	9475308159734538249013238
2796605196713610405408019	5181234096130144084041856
7426441829541573444964139	9492376623917486974923202
2931016394761975263190347	5198267398125617994391348
7632198126531809327186321	9511972558779880288252979
2933458058294405155197296	5317592940316231219758372
7712154432211912882310511	9602413424619187112552264
3075514410490975920315348	5384358126771794128356947
7858918664240262356610010	9631217114906129219461111
8149436716871371161932035	3157693105325111284321993
3111474985252793452860017	5439211712248901995423441
7898156786763212963178679	9908189853102753335981319
3145621587936120118438701	5610379826092838192760458
8147591017037573337848616	9913237476341764299813987
3148901255628881103198549	5632317555465228677676044
5692168374637019617423712	8176063831682536571306791

Figure 15.4 Ninety 25-digit numbers. Can you find two different subsets of these numbers that have the same sum?

A = all subsets of numbers

$|A| = 2^{90} > 1.238 \times 10^{27}$

B = all possible sums

$= \{0 \ldots \text{max\_sum}\}$

$\uparrow$

$90 \times 10^{25} = 0.9 \times 10^{27}$

$|B| = 0.9 \times 10^{27} + 1$

$|B| < |A|$, so two subsets must have the same sum.

but don't know which!
(nonconstructive proof).

(Thanks to Lehman, Leighton, Meyer's "Mathematics for CS" for Donuts, BOOKEEPER, hairs on heads, and subset-sum examples!)