Counting + Combinatorics

CS 330: Discrete Structures
Counting is fundamental to answering many problems that arise in discrete mathematics.

E.g., how many operations are required to carry out a given algorithm on some input? How many guesses do we need to brute force a password/key? How many different arrangements of pieces does a game-playing algorithm need to consider?
Combinatorics is a branch of mathematics that deals with counting and arranging things!

e.g., combinations, permutations, enumeration
Important: we don't typically want to enumerate all the things in order to count them!

Sometimes "counting" can use quite a bit of mathematical cleverness!
Basic combinatorial techniques:

1. Sum rule
2. Product rule
3. Subtraction rule
4. Division rule
Sum rule

If we must choose an element from either set $A$ or $B$, and $A$ and $B$ are disjoint (i.e., $A \cap B = \emptyset$), then there are $|A| + |B|$ choices in total.
e.g., if a dungeon gives us a choice to go down one of two paths A or B, and A ends in 3 doors and B ends in 5, how many different doors can we choose from?

\[3 + 5 = 8\]
Product rule

If we must choose an element from both set A and B (i.e., one from each), then there are $|A| \times |B|$ choices in total.

The choices must be independent. I.e., regardless of the element we choose in A, there are still $|B|$ choices in B, and vice versa.
e.g., a proposed license plate numbering system consists of 3 letters followed by 3 digits. How many possible license plates are there?

A: pick letter 1 — 26 choices
B: " 2 — "
C: " 3 — "
D: " digit 1 — 10 choices
E: " 2 — "
F: " 3 — "

26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000
e.g., a traveling salesman wants to visit all 50 states. How many different ways (i.e., distinguished by visitation order) can they do this?

1st state = 50 choices
2nd state = 49 choices
3rd state = 48 choices

\[ 50 \times 49 \times 48 \times \ldots \times 1 = 50! \approx 3.04 \times 10^{64} \]

Each specific ordering is called a "permutation."
e.g., how many functions are there from set $A$ to set $B$?

- for each element of $A$, we have $n_B$ choices
- we must choose independently for all values of $A$
- $n_B \times n_B \times \ldots \times n_B = n_B^{n_A}$ ways

$|A| = n_A$  $|B| = n_B$

e.g. how many one-to-one function are there from $A$ to $B$?

- for $a_1$, we have $n_B$ choices; for $a_2$, $n_B - 1$ choices, ...
- $n_B \times (n_B - 1) \times (n_B - 2) \times \ldots \times (n_B - (n_A - 1))$

# choices taken by preceding values in $A$
e.g., I want to travel to Istanbul.

There are 3 flights and 2 ships to Bordeaux.

From Bordeaux, there are 2 flights and 4 trains to Istanbul.

How many ways are there to travel to Istanbul via Bordeaux?

\[
\text{product rule} = (3 + 2) \times (2 + 4) = 5 \times 6 = 30
\]

\[
\text{sum rule}
\]

Chicago \rightarrow Bordeaux \rightarrow Istanbul
e.g., how many different top-ten rankings exist for a list of 100 songs?

\[ 100 \times 99 \times 98 \times 97 \times 96 \times \cdots \times 91 = \frac{100!}{90!} \]

An \textit{r-permutation} of set \( A \) is an ordered collection of \( r \) elements drawn from \( A \).

Given a set of \( n \) elements, and \( 1 \leq r \leq n \), how many distinct \( r \)-permutations of the set are there?

\[ P(n,r) = \frac{n!}{(n-r)!} \]

*Each permutation contains no repeated elements.*
e.g., how many different sequences of 5 rolls are there given a 6-sided die?

$$6 \times 6 \times 6 \times 6 \times 6 = 6^5$$

given a set of $n$ elements, and $1 \leq r \leq n$, how many distinct $r$-permutations of the set are there, if repetitions are allowed?

$$= n^r$$
**Subtraction rule (aka. inclusion-exclusion)**

If we must choose an element from either set $A$ or $B$, where $A$ and $B$ are overlapping (i.e., $A \cap B \neq \emptyset$), then there are $|A| + |B| - |A \cap B|$ choices in total.

$\begin{align*}
\text{total unique choices} &= |A| (6) + |B| + 5 - |A \cap B| (2) \\
&= 11 - 2 = 9
\end{align*}$
e.g., given 100 CS majors, 50 CPE majors, how many total students in these two departments?

100 + 50 = 150?

*only if sets of CS and CPE majors are disjoint sets!*
e.g., given 100 CS majors, 50 CPE majors
how many total students in these two departments?
also given: 20 CS/CPE majors

\[ |CS \cup CPE| = |CS| + |CPE| \]
\[ - |CS \cap CPE| \]
\[ = 100 + 50 - 20 = 130 \]
e.g., given 100 CS majors, 50 CPE majors, 30 EE majors,
how many total students in these three departments?

also given:  20 CS/CPE majors
           15 CS/EE majors
           10 CPE/EE majors
           5 CS/CPE/EE majors

\[ |CS \cup CPE \cup EE| = |CS| + |CPE| + |EE| \]
\[-|CS \cap CPE| - |CS \cap EE| - |CPE \cap EE| \]
\[+ |CS \cap CPE \cap EE| \]
\[= 100 + 50 + 30 - 20 - 15 - 10 + 5 = 140 \]
Division rule

If set $A$ is the union of $n$ non-overlapping sets, each containing $d$ elements, then $|A| = n \cdot d$, and $n = \frac{|A|}{d}$.

How is this useful?

$A =$ all elements under consideration

$n$ total distinct elements

$d$ elements that are effectively equivalent
e.g., how many ways are there of seating 4 guests at a round table?

\[ d = 4 \text{ equivalent ways} \]

\[ n = \frac{|A|}{d} = \frac{4!}{4} = 3! = 6 \]
e.g., how many ways are there of seating $N$ guests at a round table?

\[ \text{total ways} = \frac{N!}{N} = (N-1)! \]
e.g., how many different groups of 4 students can be formed from a 100-student class?

\[ \text{# 4-permutations} = 100 \times 99 \times 98 \times 97 \]

-but order doesn't matter within group!

\[ \text{# equivalent ways of ordering 4 students} = 4! = 24 \]

by division rule, \( \text{# different groups} = \frac{100 \times 99 \times 98 \times 97}{24} \)

an r-combination of set A is a subset of A with r elements.

how many r-combinations exist for an n-element set?

\[ C(n, r) = \binom{n}{r} = \frac{n!}{(n-r)! \ r!} \]

"n choose r" aka "binomial coefficient"
Some special cases:

\[
\begin{align*}
\binom{n}{0} &= \frac{n!}{0!} = 1 & \phi \text{ is the only size 0 subset} \\
\binom{n}{1} &= \frac{n!}{(n-1)!1!} = n & \text{n subsets of size 1} \\
\binom{n}{n} &= \frac{n!}{0!n!} = 1 & \text{1 subset of size n (the set itself)} \\
\binom{n}{n-1} &= \frac{n!}{1!(n-1)!} = n & \text{n subsets of size n-1}
\end{align*}
\]
e.g. what are the coefficients in the expansion of \((x+y)^3\) ?

\[(x+y)(x+y)(x+y)\]
\[= (x^2 + 2xy + y^2)(x+y)\]
\[= (x^3 + x^2y + 2x^2y + 2xy^2 + xy^2 + y^3)\]
\[= (x^3 + 3x^2y + 3xy^2 + y^3)\]
e.g. what are the coefficients in the expansion of \((x+y)^4\)?
Binomial Theorem

\[ \forall n \in \mathbb{N}, \quad (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \]

e.g. what are the coefficients in the expansion of \((x+y)^4\)?

\[
(x+y)^4 = \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4
\]

\[
= x^4 + 4 x^3 y + 6 x^2 y^2 + 4 x y^3 + y^4
\]
Prove \( \forall n \geq 0 \ \sum_{k=0}^{n} \binom{n}{k} = 2^n \)

1. Binomial theorem: \( 2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k} \)

2. Combinatorial: consider \# of subsets of an \( n \)-element set.

- \# of 0-element subsets = \( \binom{n}{0} \)
- \# of 1-element subsets = \( \binom{n}{1} \)
- \# of 2-element subsets = \( \binom{n}{2} \)
- \# of \( n \)-element subsets = \( \binom{n}{n} \)

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n \Rightarrow \text{i.e.} \quad 2^n = \sum_{k=0}^{n} \binom{n}{k}
\]

we also know that set of \( n \) elements has \( 2^n \) subsets
Combinatorial proof techniques:

1. **Double counting proof**: Show that two expressions $e_1$ and $e_2$ are just different ways of counting the same set $S$. Since $|S| = e_1$ and $|S| = e_2$, $e_1 = e_2$.

2. **Bijective proof**: Prove the existence of a bijective function $f: A \rightarrow B$, which shows $|A| = |B|$. If we can count $B$, then we can also count $A$. 
prove: \( \binom{n}{r} = \binom{n}{n-r} \) when \( 0 \leq r < n \)

e.g. \( \binom{6}{2} = \binom{6}{4} - \frac{6!}{(6-2)! \cdot 2!} = \frac{6!}{(6-4)! \cdot 4!} \)

\[
\binom{n}{r} = \frac{n!}{(n-r)! \cdot r!} = \frac{n!}{r! \cdot (n-r)!} = \frac{n!}{(n-(n-r))! \cdot (n-r)!} = \binom{n}{n-r}
\]

(algebraic proof)
prove: \( \binom{n}{r} = \binom{n}{n-r} \) when \( 0 \leq r < n \)

combinatorial:
- consider a set \( S \) with \( n \) elements, and the set \( R \), which contains all subsets of \( S \) with \( r \) elements; \( |R| = \binom{n}{r} \)
- the function \( f: R \to R' \), where \( f(a) = \overline{a} \), is a bijection between the subsets of \( S \) with \( r \) elements and the subsets of \( S \) with \( n-r \) elements; \( |R'| = \binom{n}{n-r} \)
- since \( f: R \to R' \) is a bijection, \( |R| = |R'| \), and \( \binom{n}{r} = \binom{n}{n-r} \)
\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}
\] for \(0 \leq k \leq n\)

Consider trying to form a team of \(k\) players from a roster of \(n = \binom{n}{k}\).

Consider trying out for a team with \(k\) spots against \(n-1\) other players.

Either:
1. You are selected, along with \(k-1\) other players = \(\binom{n-1}{k-1}\)
2. You are not selected, so \(k\) players are chosen = \(\binom{n-1}{k}\)

These are disjoint!

So:
\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}
\]
"Pascal's Triangle" identity

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \binom{1}{1} \\
\binom{2}{0} \binom{2}{1} \binom{2}{2} \\
\binom{3}{0} \binom{3}{1} \binom{3}{2} \binom{3}{3} \\
\binom{4}{0} \binom{4}{1} \binom{4}{2} \binom{4}{3} \binom{4}{4} \\
\binom{5}{0} \binom{5}{1} \binom{5}{2} \binom{5}{3} \binom{5}{4} \binom{5}{5} \\
\binom{6}{0} \binom{6}{1} \binom{6}{2} \binom{6}{3} \binom{6}{4} \binom{6}{5} \binom{6}{6} \\
\binom{7}{0} \binom{7}{1} \binom{7}{2} \binom{7}{3} \binom{7}{4} \binom{7}{5} \binom{7}{6} \binom{7}{7} \\
\binom{8}{0} \binom{8}{1} \binom{8}{2} \binom{8}{3} \binom{8}{4} \binom{8}{5} \binom{8}{6} \binom{8}{7} \binom{8}{8}
\end{array}$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$\begin{array}{cccccccc}
1 & 1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1
\end{array}$$
e.g., given $A =$ ways to select a dozen donuts from 5 varieties
$B =$ all 16-bit sequences up exactly four 1’s,

show a bijection from $A \rightarrow B$.

e.g. of $A$: 000 00 00 0 0 000
chocolate glazed strawberry plain cruller

e.g. of $B$: 000 1 00 1 00 1 0 1 0000
^ ^ ^ ^ ^

there is a bijection from $A \rightarrow B$! so $|A| = |B|$
e.g., given \( A = \) ways to select a dozen donuts from 5 varieties

\[ B = \) all 16-bit sequences w/ exactly four 1's , \]

\[
0000000000000000
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
\]

need to pick 4 bits to turn "on"

e.g., \{6,8,12,16\} \rightarrow 0 0 0 0 0 1 0 1 0 0 0 \mid 0 0 0 1

i.e., there is a bijection from the set of \( n \)-bit sequences w/ \( m \) 1's to the set of \( m \)-element subsets of an \( n \)-element set.

\# of \( n \)-bit sequences w/ \( m \) 1's = \( \binom{n}{m} \)
e.g., given $A =$ ways to select a dozen donuts from 5 varieties
$B =$ all 16-bit sequences up exactly four 1's,

$\# \text{ of } n\text{-bit sequences with } m \text{ 1's} = \binom{n}{m}$

$\# \text{ of ways to select } d \text{ donuts from } t \text{ types} = \binom{d+t-1}{t-1} = \binom{d+t-1}{d}$

$\left|A\right| = \left|B\right| = \binom{16}{4}$

by $\binom{n}{r} = \binom{n}{n-r}$
### Table 1: Combinations and Permutations With and Without Repetition

<table>
<thead>
<tr>
<th>Type</th>
<th>Repetition Allowed?</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$-permutations</td>
<td>No</td>
<td>$\frac{n!}{(n-r)!}$</td>
</tr>
<tr>
<td>$r$-combinations</td>
<td>No</td>
<td>$\frac{n!}{r! \cdot (n-r)!}$</td>
</tr>
<tr>
<td>$r$-permutations</td>
<td>Yes</td>
<td>$n^r$</td>
</tr>
<tr>
<td>$r$-combinations</td>
<td>Yes</td>
<td>$\frac{(n + r - 1)!}{r! \cdot (n - 1)!}$</td>
</tr>
</tbody>
</table>

$P(n, r) = \binom{n}{r}$

$C(n, r) = \binom{n}{r}$

$r$ donuts from $n$ types

$C(n+r-1, n-1) = \binom{n+r-1}{n-1}$
e.g., how many different strings can we make from
permuting characters in "BOOKKEEPER"?

- 10 chars, drawn from: 1 B, 20's, 2 K's, 3 E's, 1 P, 1 R
  "indistinguishable" elements

- bijection to the # of ways we map char positions to

\[
\binom{10}{1} \cdot \binom{9}{2} \cdot \binom{7}{2} \cdot \binom{5}{3} \cdot \binom{2}{1} \cdot \binom{1}{1}
\]
"permutations of indistinguishable objects"

The number of permutations of $n$ elements, given $k_1$ indistinguishable elements of type 1, $k_2$ indistinguishable elements of type 2, ..., and $k_m$ indistinguishable elements of type $m$, is:

\[
\frac{n!}{(n-k_1)!k_1!(n-k_1-k_2)!k_2!...\cdot(n-k_1-...-k_{m-1})!k_m!}.
\]

\[
= \frac{n!}{k_1!k_2!...k_m!}.
\]
e.g., how many different strings can we make from permuting characters in "BOOKKEEPER"?

\[
\frac{10!}{1!2!2!3!1!1!} = 15,1200
\]

e.g., you are guessing a video game code that consists of 2 ups, 2 downs, 2 lefts, and 2 rights. How many possible codes exist?

\[
\frac{8!}{(2!)^4} = 2,520
\]
e.g., how many socks must you pull out of a drawer containing red, green, and blue socks before you find a matching pair?
**Pigeonhole Principle**

If there are $n$ pigeons occupying $m$ holes, and $n > m$, then there must be at least one hole with two pigeons.
e.g., how many socks must you pull out of a drawer containing red, green, and blue socks before you find a matching pair?

socks = pigeons

colors = holes (3)
generalized Pigeonhole Principle:

for sets $A, B$, if $|A| > k \cdot |B|$, then every function $f: A \rightarrow B$ maps at least $k+1$ elements of $A$ to the same element of $B$. 
e.g., at least how many people in Chicago have the same # of hairs on their head? (ignoring bald people)

pop. of Chicago ≈ 2.5 million (say, > 2 million non-bald)
max # of hairs on a head = 200,000

let A be set of non-bald Chicagoans
B be \{h | 0 < h ≤ 200,000\}

f: A → B, map a person to # of hairs on their head

|A| > 10 · |B| — at least 11 people!
For any set of numbers, the A set includes all subsets of numbers, while the B set includes all possible sums. However, without further constraints, it's impossible to determine which subsets have the same sum. This is a non-constructive proof, meaning we cannot explicitly find the subsets without additional information.

| Figure 15.4 | Ninety 25-digit numbers. Can you find two different subsets of these numbers that have the same sum? |}

\[
A = \text{all subsets of numbers} \quad |A| = 2^{90} > 1.238 \times 10^{27}
\]

\[
B = \text{all possible sums} = \{0 \ldots \text{max sum} \} \\
\text{max sum} = 90 \times 10^{27} = 0.9 \times 10^{27} + 1
\]

\[
|B| > |A|, \text{ so two subsets must have the same sum}.
\]

...but don't know which!

(non-constructive proof)
Thanks to Lehman, Leighton, Meyer's "Mathematics for CS"
for Donuts, BOOKEEPER, hairs on heads, and subset-sum examples!