# Induction & Recursion

CS 330 : Discrete Structures

Many conjectures have the form: $\forall n \in \mathbb{N}\ P(n)$

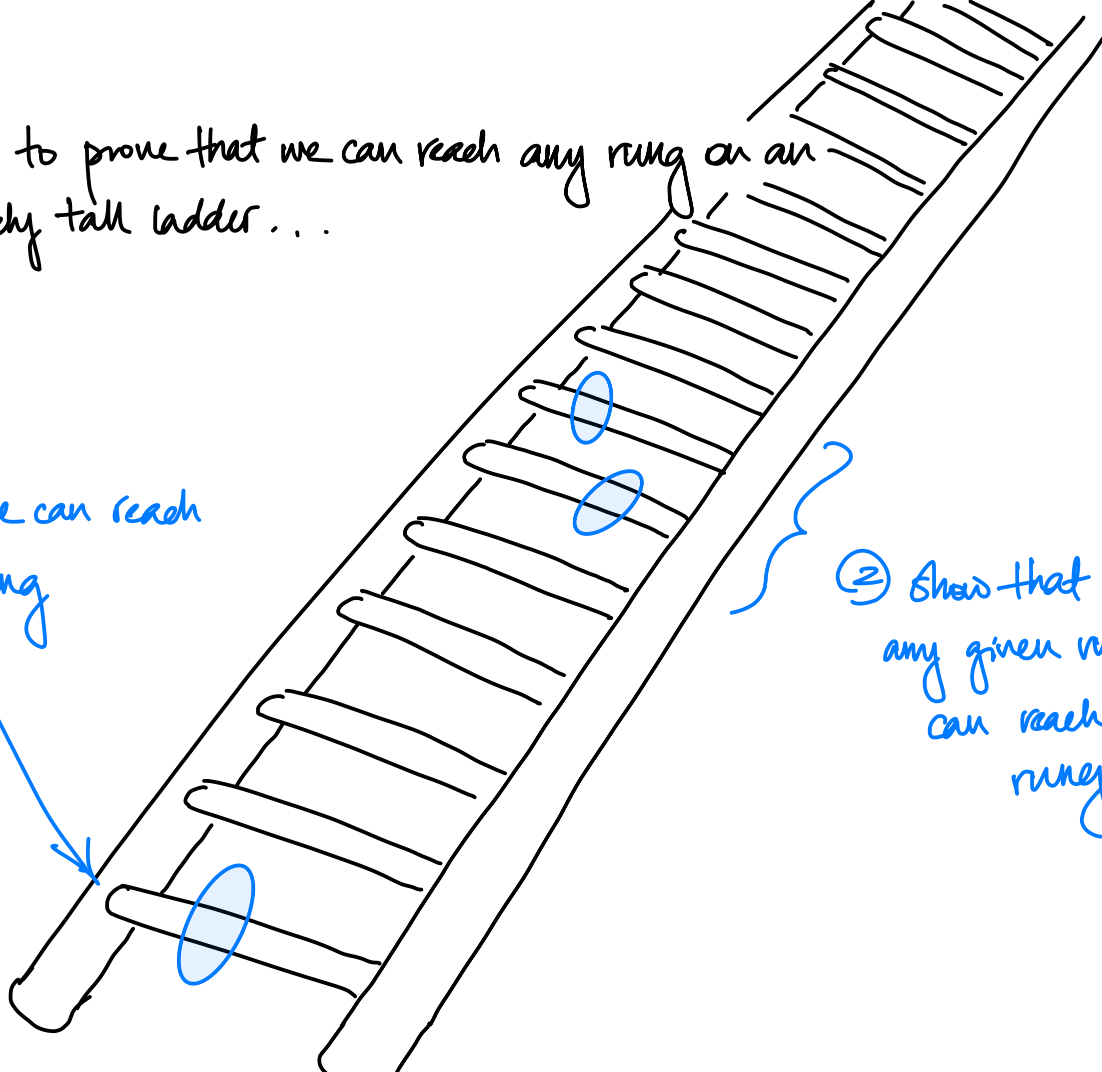
**Mathematical induction** is a technique for proving conjectures of this form based on the rule of inference:

$$\begin{array}{ll} P(1) & [\text{basis}] \\ \forall\ (P(n) \rightarrow P(n+1)) & [\text{inductive step}] \\ \hline \forall\ P(n) & [\text{proof goal}] \end{array}$$

can be for arbitrary $P(c)$, depending on domain of proof

i.e., assuming $P(n)$ is true ("inductive hypothesis"), prove $P(n+1)$ must be true

Intuition: to prove that we can reach any rung on an infinitely tall ladder...



1. show that we can reach the first rung

2. show that from any given rung, we can reach the next rung

e.g. prove that $\forall n \left(2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1\right)$

$P(n)$: $2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$

basis: $P(0)$: $2^0 = 2^{0+1} - 1$

$$1 = 2 - 1 = 1 \quad \checkmark$$

inductive step: show that $P(n) \rightarrow P(n+1)$

$$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$$

[inductive hypothesis]

surprising step! isn't that what we're trying to prove? ("circular reasoning")

add $2^{n+1}$ to both sides

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^n + 2^{n+1} &= 2^{n+1} - 1 + 2^{n+1} \\ &= 2\left(2^{n+1}\right) - 1 \\ &= 2^{n+2} - 1 \end{aligned}$$

"quod erat demonstrandum"

QED. (or $\square$)

e.g. prove that $\forall n \geq 5,\ 2^n > n^2$

$P(n): 2^n > n^2$

basis: $P(5)$: $2^5 > 5^2$

$32 > 25$ ✓

inductive step: show $P(n) \to P(n+1)$

assume $P(n): 2^n > n^2$ [inductive hypothesis]

$$
\begin{aligned}
2^{n+1} &= 2 \cdot 2^n \\
&> 2n^2 \quad [\text{by}\ \ ] \\
&= n^2 + n^2 \\
&\geq n^2 + 5n \quad [\text{since } n \geq 5] \\
&> n^2 + 2n + 1 = (n+1)^2
\end{aligned}
$$

QED.

e.g. Fibonacci sequence : $F_1 = 1$ , $F_2 = 1$ , $F_n = F_{n-1} + F_{n-2}$ for $n > 2$

1, 1, 2, 3, 5, 8, 13, 21, . . .

prove that $F_1 + F_3 + F_5 + \ldots + F_{2k-1} = F_{2k}$ for $n \geq 1$

basis : $p(1)$ : $F_1 = 1 = F_2$ $\checkmark$

inductive step: assume $F_1 + F_3 + \ldots F_{2k-1} = F_{2k}$

add $F_{2k+1}$ to each side

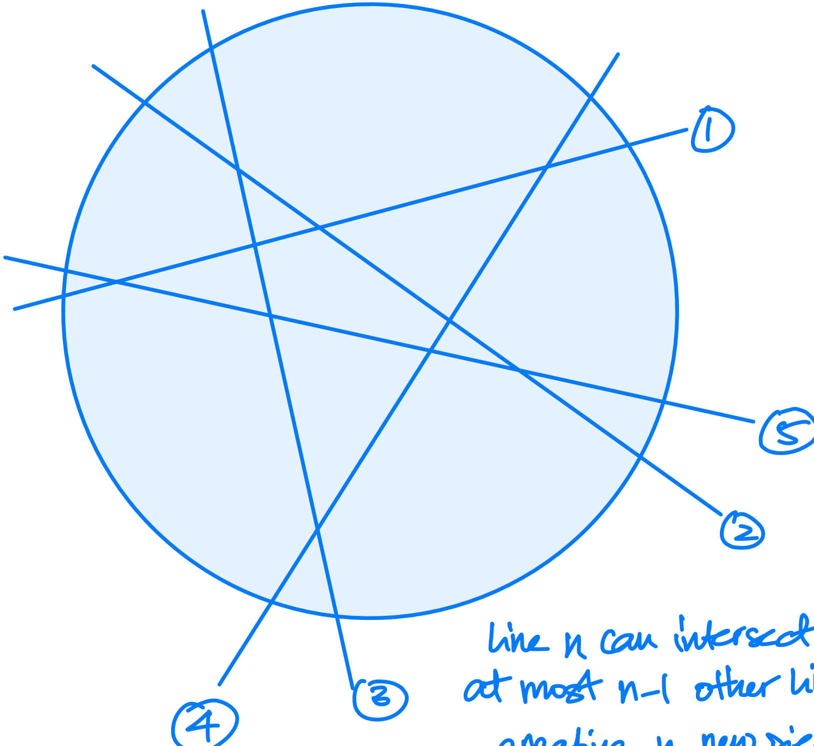
$$F_1 + F_3 + \ldots + F_{2k-1} + F_{2k+1} = F_{2k} + F_{2k+1}$$

$$= F_{2k+2}$$

$$= F_{2(k+1)}$$

QED

e.g., what is the maximum # of pieces into which we can divide a circular pizza using $n$ straight cuts?




| cuts ($n$) | additional pieces | total pieces |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 2 |
| 2 | 2 | 4 |
| 3 | 3 | 7 |
| 4 | 4 | 11 |
| 5 | 5 | 16 |

line $n$ can intersect at most $n-1$ other lines, creating $n$ new pieces

$$
\begin{aligned}
\text{pieces}(n) &= 1 + 1 + 2 + \ldots + n \\
&= 1 + \left(\frac{n+1}{2}\right) n = 1 + \frac{n^2+n}{2} \\
&= \frac{n^2+n+2}{2}
\end{aligned}
$$

e.g., prove that the maximum # of pieces into which we can divide a circular pizza using $n$ straight cuts is $\dfrac{n^2+n+2}{2}$

basis : 0 cuts $= \dfrac{0+0+2}{2} = 1$ piece ✓

inductive step: assume for $n$ cuts $= \dfrac{n^2+n+2}{2}$ pieces

for $n+1$ cuts $= \dfrac{n^2+n+2}{2} + n+1$

$$= \frac{n^2+3n+4}{2} = \frac{n^2+2n+1+n+3}{2}$$

$$= \frac{(n+1)^2+(n+1)+2}{2}$$

QED.

e.g., prove that all cats are the same size (aka cats are liquid)

$P(n)$: for any set of $n$ cats, all the cats are the same size.

basis: $P(1)$ ✓

inductive step: assume $P(n)$

$c_1, c_2, \ldots, c_n$ are the same size

$c_2, \underbrace{c_3, \ldots, c_{n+1}}_{n \text{ cats}}$ are the same size

this works for $P(2) \to P(3)$, $P(3) \to P(4)$, . . .

but what about $P(1) \to P(2)$?

~~we know size$(c_1)$ = size$(c_2)$~~

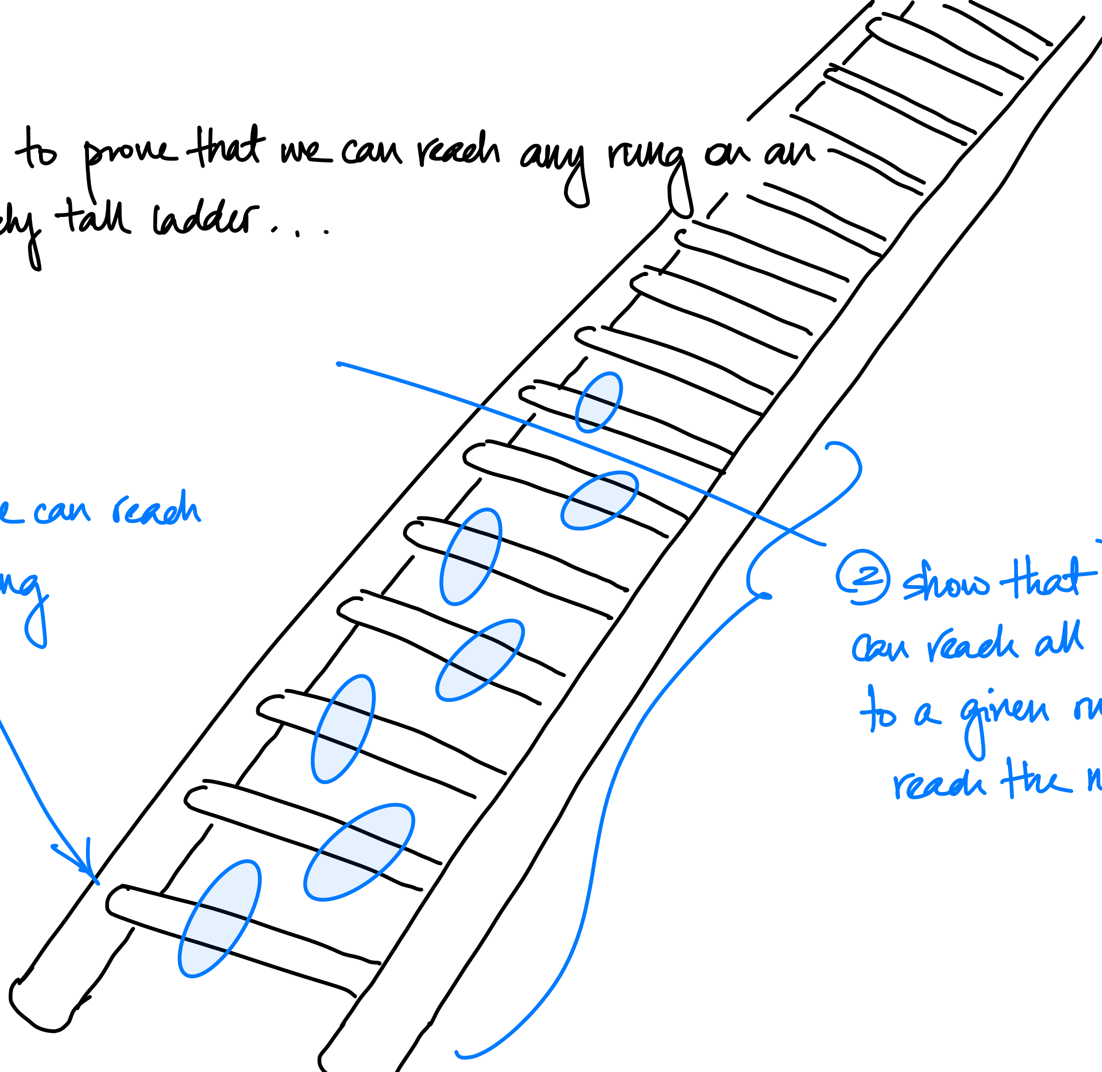
~~QED~~. (!)

— take care to flesh out all required base cases!

strong induction is a variant of mathematical induction where we prove conjectures of the same form ($\forall n \in \mathbb{N}\ P(n)$), but w/ a different hypothesis in the inductive step:

$$\frac{\begin{array}{ll} P(1) & \text{[basis]} \\ (\forall k \leq n\ P(k)) \rightarrow P(n+1) & \text{[inductive step]} \end{array}}{\forall n\ P(n) \quad \text{[proof goal]}}$$

Intuition: to prove that we can reach any rung on an infinitely tall ladder...



① show that we can reach the first rung

② show that if we can reach all rungs up to a given rung, we can reach the next rung.

e.g. every integer $n > 1$ is either prime or a product of primes.

$P(n)$ = $n$ is either prime or a product of primes

basis : $P(2)$ — 2 is prime ✓

inductive step : assume $P(k)$ is true for $2 \le k \le n$ [inductive hypothesis]

show $P(n+1)$ must hold

case 1 : $n+1$ is prime

case 2 : $n+1$ is not prime

- this means there are two integers $a, b$ where $n+1 = ab$ and $2 \le a, b \le n$
- $a, b$ are prime or product of primes (due to I.H.) so $n+1$ is prime or product of primes. QED

e.g., prove that any amount of $n \geq 8$ can be made with denominations of 3 and 5

$$8 = 3+5 \xrightarrow{-5,+6} 9 = 3+3+3 \xrightarrow{-9,+10} 10 = 5+5 \xrightarrow{-5,+6} 11 = 3+3+5 \xrightarrow{-5,+6} 12 = 3+3+3+3 \xrightarrow{-9,+10}$$

basis: $P(8)$ ✓

inductive step: show that $P(n) \rightarrow P(n+1)$

(weak induction)

assume $P(n)$; to satisfy $P(n+1)$

case 1: amount $n$ uses at least one 5

- remove 5, replace w/ two 3s

case 2: amount $n$ uses no 5s

- remove three 3s (must exist, since $n \geq 8$) replace w/ two 5s

QED.

e.g., prove that any amount of $n \geq 8$ can be made with denominations of 3 and 5

$$8 = 3+5 \xrightarrow{-5,+6} 9 = 3+3+3 \xrightarrow{-9,+10} 10 = 5+5 \xrightarrow{-5,+6} 11 = 3+3+5 \xrightarrow{-5,+6} 12 = 3+3+3+3 \xrightarrow{-9,+10}$$

basis: $P(8)$, $P(9)$, $P(10)$, $P(11)$, $P(12)$

I.H.

inductive step: assume $P(n)$ is true for $8 \leq n \leq k$, where $k > 12$

(strong induction)

to make the amount $k+1$, we can simply add 5 to $k-4$, where $P(k-4)$ is true due to the I.H.

QED.

the approach used in a proof by mathematical induction can also be used to define functions with domain $\mathbb{N}$.

e.g. $f(0) = 1$ [base case]

$f(n+1) = (n+1) \cdot f(n)$ [recursive definition]

factorial function $f(n) = n!$

Such recursively defined functions are well-defined for all values of the domain.

— induction is a great fit for proving properties of functions defined in this way!

List the first six terms in the range of $f: \mathbb{N} \to \mathbb{N}$, where

$$f(0) = 0$$

$$f(n+1) = 2 \cdot f(n) + n$$

$$f(0) = 0$$

$$f(1) = 2 \cdot 0 + 0 = 0$$

$$f(2) = 2 \cdot 0 + 1 = 1$$

$$f(3) = 2 \cdot 1 + 2 = 4$$

$$f(4) = 2 \cdot 4 + 3 = 11$$

$$f(5) = 2 \cdot 11 + 4 = 26$$

prove that this function evaluates to $2^n - n - 1$ for all $n \in \mathbb{N}$

$$f(0) = 0$$

$$f(n+1) = 2 \cdot f(n) + n$$

basis: $f(0) = 2^0 - 0 - 1 = 0$

inductive step: $f(n) = 2^n - n - 1$ [inductive hypothesis]

$$f(n+1) = 2 \cdot \left(2^n - n - 1\right) + n$$

$$= 2^{n+1} - 2n - 2 + n$$

$$= 2^{n+1} - n - 2$$

$$= 2^{n+1} - (n+1) - 1$$

QED.