Sets, Relations, and Functions

CS 330 : Discrete Structures

Sets are unordered collections of objects

- if a set is finite, we can represent it by listing all its elements in curly braces:

$$A = \{42, 9, 22, 78\}$$

- we can also specify a set by establishing a pattern and using ellipsis:

$$B = \{1, 3, 5, 7, \dots\}$$

- but this can be ambiguous, so we prefer set builder notation:

$$C = \{x \mid \underbrace{x \text{ is a prime number}}_{\text{elementhood test.}}\}$$

we use '$\in$' to express set membership.

$$\text{Michael} \in \{x \mid x \text{ is faculty at IIT}\}$$

$$\text{ঘ} \notin \{c \mid c \text{ is a letter in the English alphabet}\}$$

we can also use this in set builder notation:

$$C = \{i \mid i \text{ is a positive integer}\}$$

$$D = \{j \in C \mid 100 \le j \le 200\}$$

given predicate $P(x)$, we define its truth set over domain D:

$$T_P = \{x \in D \mid P(x)\}$$

$\emptyset$ or $\{\ \}$ denote the empty set.

important: $\{\emptyset\}$ is not the empty set!

(it is a set containing one element, which happens to be the empty set — also: $\{\{\ \}\}$)

T/F ?

$a \in \{a, b, c\}$ T

$\{a\} \in \{a, b, c\}$ F

$a \in \emptyset$ F

$\emptyset \in \{a\}$ F

$\emptyset \in \{ \quad \}$ F

$\emptyset = \{ \quad \}$ T

$2 \in \{w \mid 6 \notin \{x \mid x \text{ is divisible by } w\}\}$ F

Some fixed names we use for sets of numbers:

$\mathbb{N}$: natural numbers $\{0, 1, 2, 3, \dots\}$

$\mathbb{Z}$: integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$

$\mathbb{R}$: real numbers

$\mathbb{Q}$: rational numbers $= \{\frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$

$\mathbb{Z}^+ / \mathbb{Z}^- / \mathbb{R}^+ / \mathbb{R}^- / \mathbb{Q}^+ / \mathbb{Q}^-$ are the sets of positive/negative integers, reals, rationals

e.g., $\{x \in \mathbb{R} \mid x^2 < 9\}$

We can restrict the domain of a quantifier using sets:

e.g., universal quantification $\forall x\, P(x)$ for domain D:

$$\forall x \in D\, (P(x))$$
$$\equiv \forall x\, (x \in D \rightarrow P(x))$$

e.g. existential quantification $\exists x\, P(x)$ for domain D:

$$\exists x \in D\, (P(x))$$
$$\equiv \exists x\, (x \in D \wedge P(x))$$

sets are equal if they contain the same distinct elements.

(order doesn't matter, duplicates don't matter!)

equal or not?

$$\{1, 2, 3, 4\} = \{4, 3, 2, 1\}$$

$$\{1, 1, 2, 3, 2, 2, 4\} = \{1, 2, 3, 4\}$$

$$\{x \in \mathbb{N} \mid x < 4\} = \{x \in \mathbb{Z} \mid 0 \le x < 4\}$$

$$\{x \in \mathbb{Z} \mid x^2 < x\} = \{x \in \mathbb{R} \mid x^2 < 0\}$$

The cardinality of a finite set A, denoted $|A|$, is the number of distinct elements in A.

$$|\{0, 1, 2, 3\}| = 4$$

$$|\emptyset| = 0$$

$$|\{2, 2, 3, 1, 2, 3\}| = 3$$

$$|\{x \mid x \text{ is prime and } x < 10\}| = 4$$

a finite set is countable.

an infinite set is countable if we can find a one-to-one correspondence between its elements and the natural numbers

(CS intuition: countable if we can "index" its elements) is the set of even numbers $\{0, 2, 4, \dots\}$ countable?

$$\begin{array}{lcccccccc} \mathbb{N} = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ & \downarrow & \downarrow & \downarrow & \downarrow & & & & \\ \text{evens} = & 0 & 2 & 4 & 6 & 8 & 10 & 12 & \dots \end{array}$$

mapping function: $f(n) = 2n$

there is a one-to-one correspondence between $\mathbb{N}$ and evens.

Ans. Yes!

is $\mathbb{Z}$ (integers $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$ countable?

$\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, \ldots\}$

mapping function: $f(n) = \begin{cases} -\dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{(n+1)}{2} & \text{if } n \text{ is odd} \end{cases}$

rewrite $\mathbb{Z}$ as $\{0, 1, -1, 2, -2, 3, -3, \ldots\}$

Ans. Yes!

is $\mathbb{Q} = \{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0 \}$ countable?

qs \ ps	1	2	3	4	5	...
1	0 $1/1$	1 $2/1$	5 $3/1$	6 $4/1$	$5/1$	
2	2 $1/2$	4 $2/2$	7 $3/2$	$4/2$		
3	3 $1/3$	8 $2/3$	$3/3$	$4/3$		
4	9 $1/4$	11 $2/4$	$3/4$			
5	10 $1/5$	$2/5$				
⋮						

Ans. Yes!

is $\mathbb{R}$ (real numbers) countable?

- let's assume yes
- consider all reals between 0 and 1 :

$$
\begin{array}{lcl}
0 & \rightarrow & 0.\,a_{00}\ a_{01}\ a_{02}\ a_{03} \\
1 & \rightarrow & 0.\,a_{10}\ a_{11}\ a_{12}\ a_{13} \\
2 & \rightarrow & 0.\,a_{20}\ a_{21}\ a_{22}\ a_{23} \\
3 & \rightarrow & 0.\,a_{30}\ a_{31}\ a_{32}\ a_{33} \\
\vdots & & \qquad\qquad\qquad\qquad \ddots \\
n & \rightarrow & 1.0000\ldots
\end{array}
$$

let's pick digits on diagonal and make a new real where its digits differ from these.

E.g., $b_{ij} = (a_{ij} + 1)\ \%\ 10$

this new real is not in our list!

$\therefore$ $\mathbb{R}$ is NOT countable.

set operations, given sets A and B:

- union: $A \cup B$

  $$A \cup B = \{x \mid x \in A \vee x \in B\}$$

- intersection: $A \cap B$

  $$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

- difference: $A - B$ or $A \setminus B$

  $$A - B = \{x \mid x \in A \wedge x \notin B\}$$

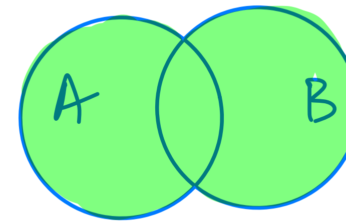
- complement: $\overline{A}$

  $$\overline{A} = \{x \in U \mid x \notin A\}$$

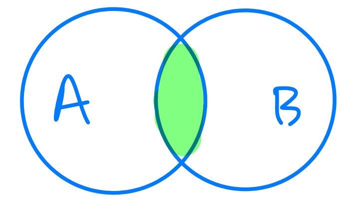
  "universal" set

Venn Diagrams

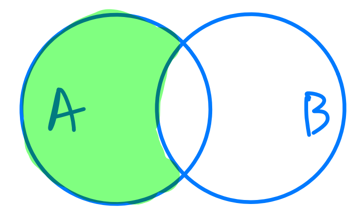
$A \cup B$



$A \cap B$



$A - B$



$\overline{A}$

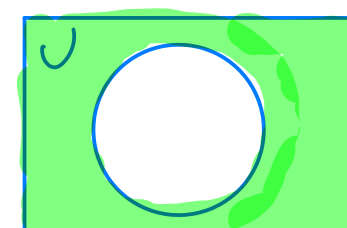


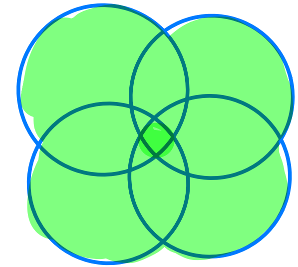
TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

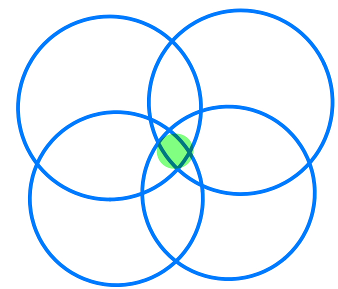
prove $\overline{A \cup B} = \overline{A} \cap \overline{B}$:

$$
\begin{aligned}
\overline{A \cup B} &= \{x \mid x \notin (A \cup B)\} \\
&= \{x \mid \neg(x \in (A \cup B))\} \\
&= \{x \mid \neg(x \in A \vee x \in B)\} \\
&= \{x \mid x \notin A \wedge x \notin B\} \\
&= \{x \mid x \in \overline{A} \wedge x \in \overline{B}\} \\
&= \{x \mid x \in (\overline{A} \cap \overline{B})\} \\
&= \overline{A} \cap \overline{B}
\end{aligned}
$$

$$A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k = \bigcup_{i=1}^{k} A_i$$



$$A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_k = \bigcap_{i=1}^{k} A_i$$



We say A is a **subset** of B ($A \subseteq B$), and B is a **superset** of A ($B \supseteq A$), iff every element in A is also in B. I.e.,

$$\forall x (x \in A \to x \in B)$$

We say A is a **proper subset** of B ($A \subset B$) iff A is a subset of B but $A \neq B$. i.e.,

$$\forall x (x \in A \to x \in B) \wedge \exists x (x \in B \wedge x \notin A)$$

the power set of S, $\mathbb{P}(S)$, is the set of all subsets of S

$$\mathbb{P}(\{a,b,c\}) = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}$$

$$\mathbb{P}(\{a\}) = \{ \emptyset, \{a\} \}$$

$$\mathbb{P}(\emptyset) = \{ \emptyset \}$$

For set S with $|S| = n$, $|\mathbb{P}(S)| = 2^n$

a	b	c
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

sets are unordered, but sometimes order matters!

an n-tuple is an ordered collection of n elements

$\underbrace{(1,2),(2,1)}_{\text{"ordered pairs"}}, (1,1,2), (2,2,1,2,1)$ are all distinct!

the Cartesian Product of two sets A and B is:

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

e.g., $\{1,2,3\} \times \{p,q\} = \{(1,p),(1,q),$
$(2,p),(2,q),$
$(3,p),(3,q)\}$

$$\{x,y,z\} \times \emptyset = \emptyset$$

$$\{x,y,z\} \times \{\emptyset\} = \{(x,\emptyset),(y,\emptyset),(z,\emptyset)\}$$

The Cartesian Product of sets $A_1, A_2, \ldots A_n$ is defined as :

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

e.g. $\{a, b\} \times \{c, d\} \times \{e, f\} = \{(a,c,e), (a,c,f), (a,d,e), (a,d,f), (b,c,e), (b,c,f), (b,d,e), (b,d,f)\}$

we also write A^2 for $A \times A$, A^3 for $A \times A \times A$, i.e.,

$$A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } i = 1, 2, \ldots, n\}$$

e.g. $\{a, b, c\}^2 = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)\}$

Note: $A \times B \neq B \times A$ (unless A or B is $\emptyset$),

and $A \times B \times C \neq (A \times B) \times C$!

e.g. $\underbrace{\left(\{a,b\} \times \{c,d\}\right)} \times \{e,f\}$

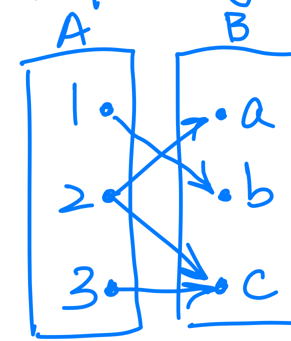
$$= \{(a,c),(a,d),(b,c),(b,d)\} \times \{e,f\}$$

$$= \{((a,c),e),((a,c),f),((a,d),e),((a,d),f),$$
$$((b,c),e),((b,c),f),((b,d),e),((b,d),f)\}$$

technically: a binary relation

A relation from set A to set B is a subset of $A \times B$

e.g., given $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, graphically:



write down a relation from A to B:

$$G = \{(1,b), (3,c), (2,a), (2,c)\}$$

- 3 is related to c through G: $3 \, R \, c$
- 3 is not related to b through G: $3 \not R \, b$

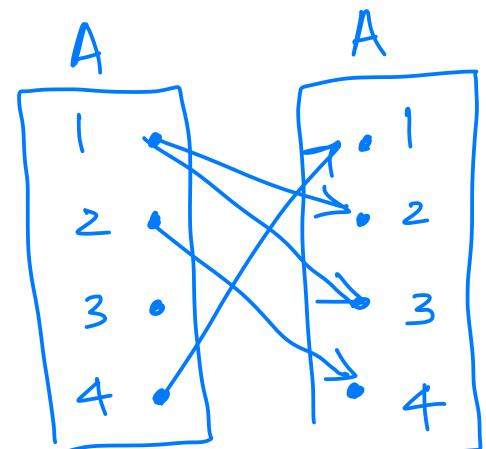
R	a	b	c
1		x	
2	x		x
3			x

A relation on a set A is a relation from A to A

(i.e., a subset of $A \times A$)

e.g., given $A = \{1, 2, 3, 4\}$, write down a relation on A

$H = \{(1,2), (1,3), (2,4), (4,1)\}$



Matrix representation

We can represent a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$ as the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

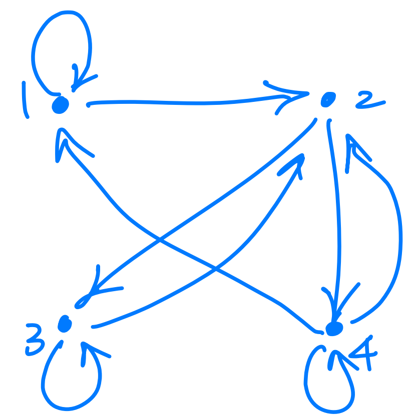
e.g., given $A = \{a, b, c\}$, $B = \{1, 2\}$ and $R = \{(a,1), (a,2), (b,2), (c,1)\}$, draw the matrix for R.

$$M_R = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

Digraph representation

We can represent a relation on a set as a directed graph (digraph), where vertices correspond to elements and edges are drawn between vertices for all ordered pair in the relation.

e.g., given $A = \{1, 2, 3, 4\}$ and relation $R = \{(1,1), (1,2), (2,3), (2,4), (3,2), (3,3), (4,1), (4,2), (4,4)\}$, draw the digraph for R

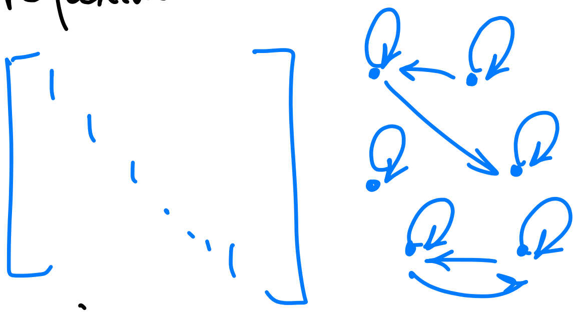


Relation R on a set A is ... (where $a, b, c \in A$)

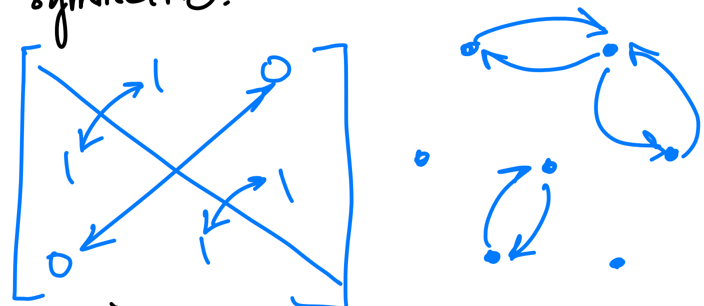
- **reflexive** if $\forall a \, ((a,a) \in R)$
- **symmetric** if $\forall a \, \forall b \, ((a,b) \in R \rightarrow (b,a) \in R)$
- **antisymmetric** if $\forall a \, \forall b \, (((a,b) \in R \wedge (b,a) \in R) \rightarrow (a=b))$
- **transitive** if $\forall a \, \forall b \, \forall c \, (((a,b) \in R \wedge (b,c) \in R) \rightarrow (a,c) \in R)$

What do matrix & digraph representations of reflexive, symmetric, antisymmetric, and transitive relations look like?

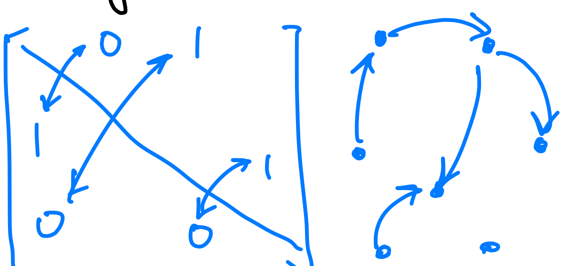
reflexive:



symmetric:

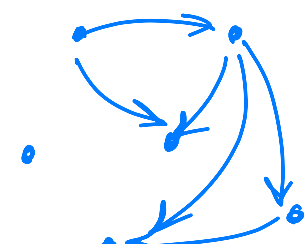


antisymmetric:



transitive:

$$\left[\begin{matrix} ? \\ \text{(examine } M_R^2\text{)} \end{matrix}\right]$$



A relation on a set is an equivalence relation if it is simultaneously reflexive, symmetric, and transitive.

Two elements a and b related by an equivalence relation are called equivalent ($a \sim b$)

e.g., consider the relation:

$$R = \{(a,b) \mid a, b \text{ are strings of English letters, and } aRb \text{ iff } \mathrm{length}(a) = \mathrm{length}(b)\}$$

is R an equivalence relation? Yes!

- reflexivity: $\mathrm{length}(a) = \mathrm{length}(a)$ ✓
- symmetry: $(\mathrm{len}(a) = \mathrm{len}(b)) \rightarrow (\mathrm{len}(b) = \mathrm{len}(a))$ ✓
- transitivity: $((\mathrm{len}(a) = \mathrm{len}(b)) \wedge (\mathrm{len}(b) = \mathrm{len}(c))) \rightarrow (\mathrm{len}(a) = \mathrm{len}(c))$ ✓

e.g., consider the relation:

"congruence mod N": $a \equiv b \pmod{N}$, which is true iff $a-b$ is evenly divisible by N

$$R = \{(a,b) \mid \overbrace{a \bmod N = b \bmod N}\} \text{ for } a,b \in \mathbb{Z}$$

and constant $N \in \mathbb{Z}^+$

is R an equivalence relation? Yes!

- reflexivity: $a \equiv a \pmod{N}$, $0 = kN$ ✓
- symmetry: $a \equiv b \pmod{N} \rightarrow a-b = kN$

  $b-a = -kN \therefore b \equiv a \bmod N$ ✓
- transitivity: $a \equiv b \pmod{N}$ and $b \equiv c \pmod{N}$

  $\rightarrow a-b = kN$ and $b-c = lN$

  $a-b+b-c = kN + lN$, $a-c = (k+l)N$

  $\therefore a \equiv c \pmod{N}$ ✓

If R is an equivalence relation on set A, the equivalence class of $a \in A$, denoted $[a]_R$, is the set $\{e \mid (a,e), \in R\}$

$\forall b (b \in [a]_R)$, b is a representative of $[a]_R$

e.g., what are the equivalence classes of 0, 1, 2 for the relation $R = \{(a,b) \mid a \equiv b \pmod{3}\}$, $a, b \in \mathbb{Z}$?

$[0]_R = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$

$[1]_R = \{\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\}$

$[2]_R = \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\}$

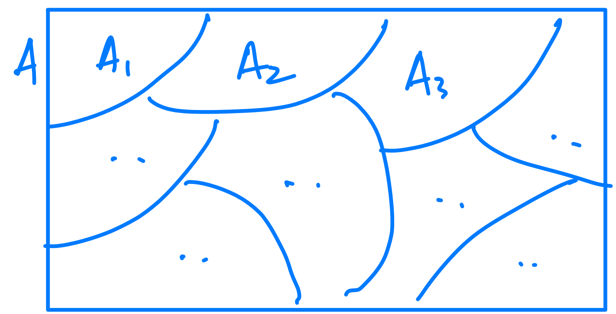
An equivalence relation R on set A partitions set A

i.e., - the equivalence classes of distinct elements a, b are either equal or disjoint :

$$[a]_R \neq [b]_R \leftrightarrow [a]_R \cap [b]_R = \emptyset$$

- the union of all equivalence classes of R is A :

$$\bigcup_{a \in A} [a]_R = A$$



disjoint subsets

A **function** f from set A to set B is a relation from A to B that assigns exactly one element of B to each element of A.

- we write $f: A \to B$ (f maps set A to set B) and $f(a) = b$ (f assigns $b \in B$ to $a \in A$)

- A is the **domain** of f, and B is the **codomain**

- The set of all elements of B assigned by f is called the **range**

types of functions

one-to-one / injection

no element of codomain is assigned to more than one value in domain

onto / surjection

all elements of codomain are assigned

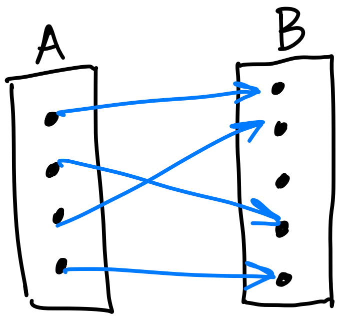
one-to-one correspondence / bijection

both one-to-one and onto

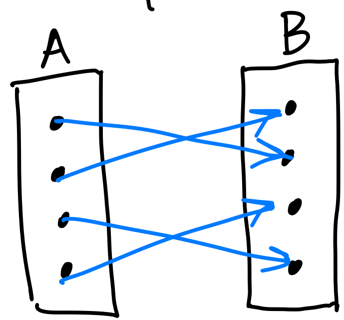
partial

not all values of domain have an assignment (i.e., they may be undefined) e.g., functions from A to B :

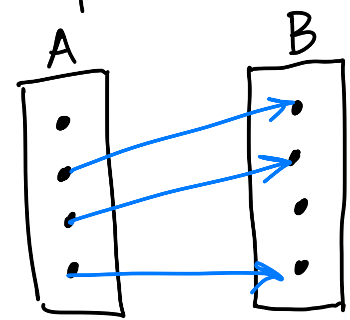
injection & not surjection



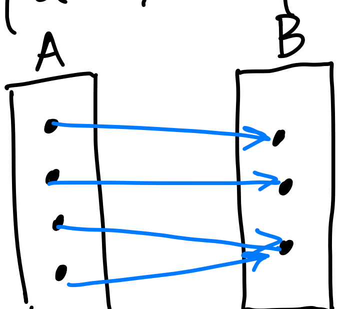
bijection



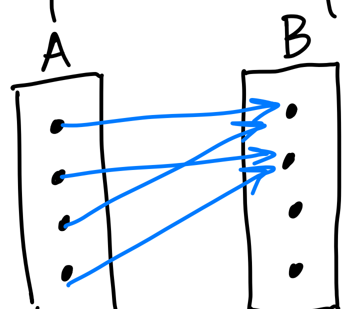
partial function



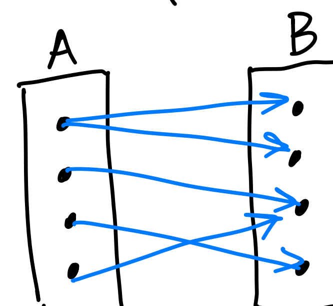
surjection & not injection



not injection & not surjection



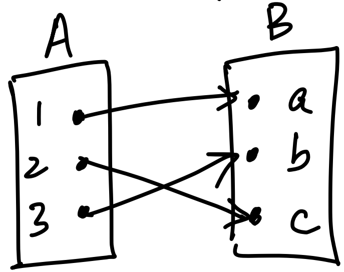
not a function



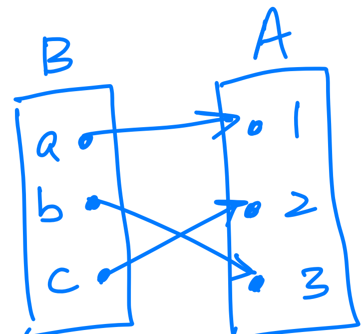
If a function f is a bijection, we can define its inverse, f^{-1}, where $f^{-1}(b) = a$ for every $f(a) = b$

e.g. draw the inverse of

f



f^{-1}



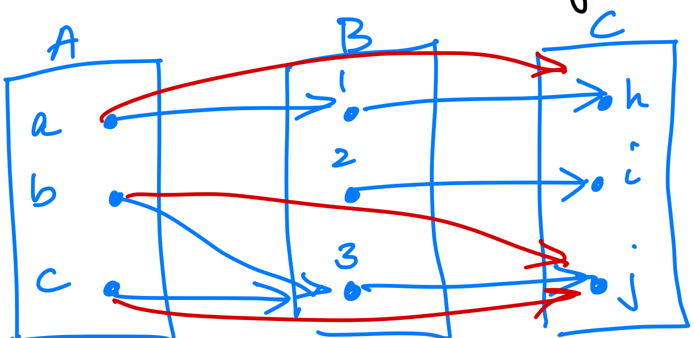
e.g., is the function $f(x) = 2x,\ x \in \mathbb{R}$ invertible? Yes

e.g., is the function $f(x) = x^2 + 1,\ x \in \mathbb{R}$ invertible? No

The composition of $f: A \to B$ and $g: B \to C$, is defined as the function $(g \circ f): A \to C$ where, for all $a \in A$,

$$(g \circ f)(a) = g(f(a))$$

e.g., given $A = \{a, b, c\}$, $B = \{1, 2, 3\}$, $C = \{h, i, j\}$,
$f = \{(a,1), (b,3), (c,3)\}$, $g = \{(1,h), (2,i), (3,j)\}$,
write down the function $g \circ f$



$g \circ f = \{(a,h), (b,j), (c,j)\}$

e.g., given functions $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$, where

$f(x) = 2x + 1$, $g(x) = 4x^2$,

find $f \circ g$ and $g \circ x$

$$(f \circ g)(x) = f(g(x)) = 2(4x^2)+1 = 8x^2+1$$

$$(g \circ f)(x) = g(f(x)) = 4(2x+1)^2 = 4(4x^2+4x+1)$$

$$= 16x^2+16x+4$$

Relationships between more than 2 sets can be expressed using n-ary relations (aka. relations of degree n)

given sets $A_1, A_2, \ldots, A_n$, an n-ary relation on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$ (which, recall, is a set of n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$)

e.g., give a relation of degree 4 on the set $\mathbb{N}$.

i.e., $R \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$

$R = \{(0,0,0,0), (0,1,3,100), (2,1,4,3)\}$

e.g., give a relation R on $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, where each triple $(a, b, c) \in R$ iff $a < b + c$

$R = \{(0, 1, 2), (100, 80, 60), (20, -20, 50)\}$

e.g., describe an n-ary relation that can be used to represent airport flight information

A : airport codes $= \{$ORD, MDW, JFK, LAX, ...$\}$

B : airline codes $= \{$AA, DL, SW, ...$\}$

C : flight numbers $= \mathbb{N}$

D : time $= \{0{:}00, 0{:}01, \ldots, 23{:}59\}$

E : remark $= \{$landed, departed, boarding, ...$\}$

relation $R \subseteq A \times B \times C \times B \times C \times D \times E$

Relational databases are conceptually built on top of storing all data as tuples grouped by relation, managed by operations defined using predicate logic.

A relational database organizes data into relations over groups of attributes, and supports operations defined using predicate logic to query and manage that data.

E.g., contact database:

first name (F)	last name (L)	email (E)
John	Doe	jdoe@a.com
Mary	Jane	mjane@b.com
⋮	⋮	⋮

$R \subseteq F \times L \times E$

{ (John, Doe, jdoe@a.com), (Mary, Jane, mjane@b.com), ... }

"table" = relation

"attribute/column" = domain

"row/record" = tuple

A domain of a relation is called the primary key when the value from this domain is unique across all tuples.

e.g.,

sid	name	email	phone
⋮	⋮	⋮	⋮

When the value from a combination of multiple domains is unique across tuples, these domains are collectively a composite key.

e.g.,

dept	course #	title
CS	330	Disc. Str.
CS	331	Data Str.

Database operations:

- selection: $\sigma_C(R)$ – selects all tuples from R for which predicate C is true

- projection: $\Pi_{a_1, a_2, \ldots, a_n}(R)$ – evaluates to the tuples from R, w/ only values from the attributes $a_1, a_2, \ldots a_n$ included

- join: $\bowtie(R, S)$ – evaluates to combined tuples from R and S, based on common attributes.

Database + SQL Demo