Propositional Logic

CS 330 : Discrete Structures

"proposition" : a declarative sentence that is either True (T / 1) or False (F / 0 / $\perp$)

"bottom" / "falsum" / "absurdum"

is it a proposition?

"This morning was sunny."

"It will not rain tomorrow."

"How are you?"

"I am sitting in my chair and typing on my laptop."

"Jump over this fence or I will cry."

propositional variables (e.g., "p", "q") stand in for propositions, and help us focus on the logic (rather than the propositions themselves)

we often prefer to use variables to refer to atomic propositions, which cannot be expressed in terms of simpler propositions.

we can qualify or combine propositions w/ logical operators:

- negation: $\neg p$ —— "not p"
- conjunction: $p \wedge q$ —— "p and q"
- disjunction: $p \vee q$ —— "p or q" (inclusive or)
- exclusive or: $p \oplus q$ —— "p xor q"

in order of decreasing precedence.

disjunction is the typical programming language "OR"

e.g.

```
def foo(x, y):
    if x < 0 OR y < 0:
        raise Exception("no negative inputs")
```

exclusive or is often implied w/ the English "OR"

e.g. "you can have cake OR ice cream for dessert."

we use truth tables to show the value of a proposition for all combinations of values taken by its variables.

e.g. $\neg p$

p	$\neg p$
T	F
F	T

$p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

$p \oplus q$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

how many rows in a truth table for a proposition of N variables? $= 2^N$

e.g. $N = 2$

p	q
T	T
T	F
F	T
F	F

4

e.g. $N = 3$

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

8

e.g. $N = 4$

p	q	r	s
1	1	1	1
1	1	1	0
1	1	0	1
1	1	0	0
1	0	1	1
1	0	1	0
1	0	0	1
		⋮	
0	0	0	0

$2^4 = 16$

Applying logic to English propositions:

p : "I love cats"

q : "computer science is a science"

r : "$4 < 100$"

read:

$p \wedge q$

$\neg p \vee \neg q$

$p \wedge (q \vee \neg r)$

$p \wedge \neg p$

$q \vee \neg q$

A tautology is a proposition that is always true.

e.g. T

$p \lor \neg p$

$(p \land q) \lor \neg p \lor \neg q$

A contradiction is a proposition that is always false.

e.g. F

$p \land \neg p$

$\neg((p \land q) \lor \neg p \lor \neg q)$

How to prove $(p \wedge q) \vee \neg p \vee \neg q$ is a tautology?

p	q	$p \wedge q$	$\neg p$	$\neg q$	$(p \wedge q) \vee \neg p \vee \neg q$
T	T	T	F	F	T
T	F	F	F	T	T
F	T	F	T	F	T
F	F	F	T	T	T

$p \to q$ is a proposition known as a conditional statement (aka. implication)

read "if p, then q"

hypothesis / antecedent

conclusion / consequent

caution: the logic conditional statement is <u>NOT</u> equivalent to the "if" statement in imperative programming!

e.g.

```
if cond {
    ...
}
```

not a proposition!

e.g. "if the AC is on, then I'll be cold."

p : the AC is on

q : I'll be cold.

truth table for $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

the hypothesis is true, but the conclusion is wrong!

can be cold for another reason

many other ways to express conditional in English.

some tricky ones for $p \to q$:

"p is sufficient for q"

"p only if q"

"q is necessary for p"

"q unless $\neg p$"

Can we express '$\rightarrow$' using just $\neg, \wedge, \vee$?

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

if $\neg p$, $p \rightarrow q$ is T

otherwise $p \rightarrow q$ is equivalent to

so, $p \rightarrow q \equiv \neg p \vee q$

"equivalent to"

contrapositive of $p \to q$ is $\neg q \to \neg p$

converse of $p \to q$ is $q \to p$

inverse of $p \to q$ is $\neg p \to \neg q$

"If the AC is on, then I'll be cold"

contrapositive:

"if I'm not cold, then the AC isn't on"

converse:

"if I'm cold, then the AC is on"

inverse:

"if the AC is not on, then I won't be cold."

prove: an implication and its contrapositive are logically equivalent.

i.e., $p \to q \equiv \neg q \to \neg p$

p	q	$p \to q$	$\neg p$	$\neg q$	$\neg q \to \neg p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

$p \leftrightarrow q$ expresses the biconditional statement "p if and only if q", which is equivalent to the proposition $(p \to q) \wedge (q \to p)$

truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

The biconditional is often implied by the English "if".

E.g., "if you brush your teeth, then you can go to bed."

(if the biconditional is not implied, what loophole exists?)

When propositions are logically equivalent, we can substitute one for the other in statements they appear in — this can be useful in many ways (e.g. simplification/reduction, standardization)

$p \equiv q$ (p is equivalent to q) if $p \leftrightarrow q$ is a tautology.

Show that $\underbrace{\neg(p \vee q)}^{r} \equiv \underbrace{\neg p \wedge \neg q}^{s}$

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$r \leftrightarrow s$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \land \mathbf{T} \equiv p$ $p \lor \mathbf{F} \equiv p$	Identity laws
$p \lor \mathbf{T} \equiv \mathbf{T}$ $p \land \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \lor p \equiv p$ $p \land p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \lor q \equiv q \lor p$ $p \land q \equiv q \land p$	Commutative laws
$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \land q) \equiv \neg p \lor \neg q$ $\neg(p \lor q) \equiv \neg p \land \neg q$	De Morgan's laws
$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv \mathbf{T}$ $p \land \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \lor q$

$p \to q \equiv \neg q \to \neg p$

$p \lor q \equiv \neg p \to q$

$p \land q \equiv \neg(p \to \neg q)$

$\neg(p \to q) \equiv p \land \neg q$

$(p \to q) \land (p \to r) \equiv p \to (q \land r)$

$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$

$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$

$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Note that the propositional variables listed in the preceding tables can stand for compound propositions.

E.g. De Morgan's Law: $\neg(p \wedge q) \equiv \neg p \vee \neg q$

$$\neg((r \wedge \neg s) \wedge (\neg t \rightarrow u)) \equiv \neg(r \wedge \neg s) \vee \neg(\neg t \rightarrow u)$$